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Comparative Analysis of Classical and Fractional-Order Models for Rabies Transmission

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Abstract: Rabies is still a serious public health problem globally, especially where there is high dog-to-human contact and low vaccination coverage. In this paper, a fractional-order mathematical model is developed to explain the transmission dynamics of rabies in dogs and humans. The model is established by adopting the Caputo-Fabrizio fractional-order derivative (CFFROD), which suits the memory effects and non-locality properties of disease progression. The model has compartments for susceptible, exposed, infected, and recovered members of both species, as well as the viral load in the environment. Existence and uniqueness of solutions are proven via fixed-point theory to ensure mathematical consistency of the model. Numerical computations via the Adams-Bashforth method are performed to analyse the dynamics of the system for a range of fractional orders. Numerical computations provide evidence that fractional-order dynamics have a considerable impact on disease progression, ensuring the significance of memory in infectious disease modelling. Based on verified experimental data, a comparison between the fractional-order and classical models is presented. The results show that the fractional model provides greater insight into transmission and control timing patterns and best fits real-world data. This study supports the use of fractional modelling in the well-informed creation of successful rabies prevention initiatives and improved comprehension of disease dynamics.

Keywords: Rabies Transmission; Fractional Order Derivative (FROD); Caputo-Fabrizio Fractional Derivative (CFFROD); Disease Dynamics; Fixed Point Theory; Numerical Simulation; Adams-Bashforth Method; Susceptible-Exposed-Infected-Recovered (SEIR) Model

1. Introduction

Rabies, a viral disease, has a long, harmful history, causing fatal effects on human and animal populations. Rabies poses a global public health challenge, thus requires an understanding of features (e.g., infection history, transmission, clinical signs), and a collaborative, multidisciplinary approach for prevention and management.

There are historical references to "mad dogs" in visual art as early as 2000 BCE in Mesopotamia, and narratives about the dangers of animal bites, which signal a basic understanding of rabies in ancient texts. Rabies has long been embedded in human fears, myths, and superstitions and has played a concretive role in societies throughout history. Transmission is via infected animal saliva in a bite or scratch. The rabies virus travels via

peripheral nerves to the central nervous system once inside the body [1–5]. Rabies causes neurological dysfunction through severe hospitalizations upon infection and concludes when support-oriented health progression results in death.

Domestic dogs are the major rabies reservoir and source of human rabies. Wild animals (e.g., foxes, raccoons, and bats) as rabies reservoirs have highlighted zoonotic characteristics of rabies in a clinical context. Rabies exists in 3 distinct phases: the prodromal phase (i.e., fever, fatigue, and headache), the furious phase (i.e., agitation, aggression, hallucinations, and hydrophobia) and the paralytic phase (e.g., muscle weakness, impaired respiratory function), which is terminal without treatment unless relieved by an unexpected trajectory [6–11].

Controlling rabies continues to be challenging because of rabies's high case fatality rate, complicated modes of transmission, and heterogeneous global distribution of rabies [12]. Low- and middle-income nations, e.g., LMIC, bear a disproportionate burden of rabies despite advances in rabies control, as they often lack access to vaccination and post-exposure prophylaxis (PEP). Rabies control measures consist of (1) dog vaccination campaigns, (2) enhanced surveillance/diagnostics, and (3) community awareness /education to promote responsible dog ownership and ultimately prevent dog bites [13–17].

Recently, new advances in rabies treatment have been introduced, which may offer options beyond traditional post-exposure prophylaxis (PEP). Monoclonal antibodies (Rabishield), which effectively neutralise the virus, could provide PEP without the unwanted adverse effects compared to traditional immunoglobulins. New adjunct therapies using modified antioxidants are also under development, which have been tested and exhibit a neuroprotective effect to reduce viral-induced oxidative stress that may ultimately impact patient survival [18,19]. Revised guidelines of rabies control by the World Health Organisation (2023) and the Centre for Disease Control (2024) recommend a rabies control framework to utilise new advances in PEP and rabies therapy [20,21].

Rabies is virtually always lethal when clinical signs appear, and domestic dogs account for up to 99 percent of rabies-related human deaths globally [22,23]. Rabies can circulate in both wild and domestic species and is commonly transmitted to humans via bites, scratches, or through loss of integrity of mucous membranes. Young children aged 5-14 are the most vulnerable. The incubation period typically lasts between 2–4 months, but can vary from a few days to more than a year, and is based primarily on the amount of exposure and the site of exposure. By the time clinical signs appear, it is exceedingly rare to survive because of viral-induced encephalitis [24,25].

Public health messaging regarding dog behaviour and dog bite prevention strategies is essential for dog vaccine campaigns. Since rabies is practically eliminated by avoidance of dog bites, there is a cost advantage of the rabies vaccine as a preventative measure over PEP. Rabies is completely preventable with effective vaccines; the most significant risk factor for getting rabies is contact and exposure to rabies in dogs. Vaccinating dogs also expands the number of humans who do not require PEP. If there is concern that a person is exposed to rabies, PEP is available in Canada. However, pre-exposure prophylaxis could be beneficial for specific high-risk groups such as veterinarians, animal control, wildlife workers, and lab personnel [26–29].

Mathematical modeling enables researchers to understand better infectious disease dynamics and potential control strategies [30–32]. In this study, we develop a fractional-order mathematical model for rabies transmission in dogs and humans based on the Caputo–Fabrizio fractional derivative (CFFROD) that incorporates memory and non-locality in transmission. In addition, the fractional-order rabies model is then accordingly solved using the numerical two-step Adams–Bashforth method, allowing for the assessment of the susceptible, exposed, infected, and recovered populations, at fractional orders . Comparing our results with a classical model provides useful insights to aid in our knowledge and understanding of rabies control [33–36].

2. The Definition and Basic Principles

Basic definitions, theorems, and conclusions for the FROD of Riemann-Liouville and Caputo-Fabrizio are covered in this section. These derivatives are widely used in the construction of fractional-order mathematical models.

2.1. Definition 1

For every arbitrary real order $\Phi > 0$, Riemann-Liouville defines the integrability of a function $f(\varpi)$ as stated in Equation (1).

$$D_{0,\varpi}^{\Phi} f(\varpi) = \frac{1}{\Gamma(\Phi)} \int_0^{\varpi} (\phi - t)^{\Phi-1} f(t) dt. \quad (1)$$

$\varpi > 0$ and $\Gamma(\Phi)$ denote the Gamma function evaluated at Φ. Using the Riemann-Liouville sense, the integral operator $D_{0,\varpi}^{\Phi}$ represents the fractional derivative of $f(\varpi)$ with respect to ϖ across the interval $[0, \varpi]$.

2.2. Definition 2

Any absolutely continuous function $f(\varpi)$, where $f(\varpi) \in C^n[0, \varpi]$ with $n > 0$, has FROD that may be expressed as the following integral [Equation (2)].

$$^{CF}D_{0,\varpi}^{\Phi}[f(\varpi)] = \frac{1}{\Gamma(\mathrm{n}-\Phi)}\int_0^{\varpi}(\varpi - t)^{\mathrm{n}-\Phi-1}f^n(t)dt. \tag{2}$$

Where $n - 1 < \Phi \leq \mathrm{n}, \mathrm{n} \in \mathrm{N}$. if $\Phi \longrightarrow 1$ then $^{CF}D_{0,\varpi}^{\Phi}f(\phi) \longrightarrow f'(t)$.

2.3. Definition 3

The CFFROD of a function $f(\varpi)$ with an order greater than zero is expressed as stated in Equation (3).

$$^{CF}D_{0,\varpi}^{\Phi}[f(\phi)] = \frac{\Psi(\Phi)}{(1-\Phi)}\int_0^{\varpi} f'(t)exp\left[-\Phi\frac{\varpi - t}{1-\Phi}\right]dt. \tag{3}$$

The function $\Psi(\Phi)$ is referred to as a normalization function, Satisfying $\Psi(0) = \Psi(1) = 1$, and f belongs to the space $H^1[0, \varpi]$. For $\mathrm{T} > 0$.

2.4. Definition 4

For a given function f, the fractional integral is expressed as stated in Equation (4).

$$D_t^{\Phi}[f(t)] = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}f(t) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^{t} f(\Phi)d\Phi, \qquad \mathrm{t} \geq 0. \tag{4}$$

Φ Represents the order of the fractional integral, Such that $0 < \Phi < 1$.

Theorem 1. *A function g(y) that is continuous on the interval [0, T] and whose (CFFROD)* $^{CF}D_{0,\varpi}^{\Phi}[g(\varpi)]$ *is finite for* $\phi \in (0, T]$ *has an existential value at a point s in the interval* $[0, \varpi]$ [Equation (5)] [31].

$$g(\phi) = g(0) + \frac{1}{\Gamma(\Phi)}\left[^{CF}D_{0,\varpi}^{\Phi}g\right](S)\varpi^{\Phi}, \tag{5}$$

Where $\Gamma(\Phi)$ is the gamma function, $0 \leq s \leq \varpi$, and $\forall\varpi \in (0, T]$.

3. Creation of the Mathematical Framework

3.1. Familiar Mathematical Model

The transmission model for rabies in dogs and people divides the population into eight divisions to represent the disease's various phases [Equation (6)]. Specifically: S_1 depicts the susceptible dog population, which includes those who are at risk of developing rabies if exposed to the virus. E_1 represents the exposed dog population, which includes dogs that have been infected but are not yet symptomatic or infectious. I_1 refers to the infected dog population, which includes dogs that have exhibited rabies symptoms and can spread the virus to others. R_1 indicates the recovered dog population, which includes dogs who have conquered the sickness and are no longer spreadable.

Likewise, for humans: S_2 refers to the susceptible individuals who are at risk of developing rabies if exposed to diseased animals. E_2 signifies the exposed human population, which includes people who have been in contact with rabid animals but have not yet shown symptoms. I_2 represents the infected individuals or those who have developed symptoms of rabies and may spread the virus to others. R_2 indicates the recovered human population, meaning people who have fully recovered from rabies and are no longer contagious.

$$\frac{dS_1}{dt} = \pi_1 - (\mu + \alpha_1 + \gamma_1)S_1 + \lambda_1 R_1$$

$$\frac{dE_1}{dt} = \alpha_1 S_1 - (\mu + \beta_1)E_1$$

$$\frac{dI_1}{dt} = \beta_1 E_1 - (\mu + \mu_1 + \delta_1)I_1$$

$$\frac{dR_1}{dt} = \gamma_1 S_1 - (\lambda_1 + \mu)R_1$$

$$\frac{dS_2}{dt} = \pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu + \tau_2 + \eta_2)S_2 \quad (6)$$

$$\frac{dE_2}{dt} = \eta_2 S_2 - (\omega_2 + \mu)E_2$$

$$\frac{dI_2}{dt} = \omega_2 E_2 - (\mu + \mu_2)I_2$$

$$\frac{dR_2}{dt} = \tau_2 S_2 - (\mu + \theta_2)R_2$$

In the transmission model for rabies in both canines and people, numerous elements play critical roles in influencing the dynamics of disease spread: α_1 indicates the rate at which susceptible canines are exposed to the virus. This parameter impacts the risk that vulnerable canines may get rabies when exposed. β_1 represents the infection rate for exposed dogs. It influences how soon exposed dogs become infected and develop rabies symptoms. μ_1 represents the disease-related mortality rate among dogs. This metric represents the death rate of rabies-infected dogs. μ_2 reflects the disease-related mortality rate in people. It represents the fatality rate among infected persons owing to rabies. μ represents the natural death rate throughout all phases. This metric accounts for death rates in both canine and human populations that are independent of rabies infection. δ_1 is the rate of transmission from canines to vulnerable people. It measures how well the rabies virus transmits from infected canines to vulnerable people. τ_2 represents the rate of healing from dog bites in vulnerable individuals. This metric represents the recovery rate of individuals who have been bitten by rabies-carrying canines but have yet to display symptoms. θ_2 reflects the rate at which recovered persons revert to being susceptible.η_2 represents the rate of exposure for sensitive persons. It assesses the chance of susceptible persons contracting the rabies virus. ω_2 represents the rate of infection among exposed persons. This parameter controls the development of rabies infection in persons who have been exposed to the infection.

The rabies transmission model relies heavily on these characteristics, as well as population numbers, vaccination rates, rates of recovery, and transmission rates for both canine and human populations. They together determine the dynamics of rabies transmission, the impact on various groups, and the efficacy of treatments and control measures in minimizing the disease's effects.

3.2. Mathematical Model for Rabies Transmission in Both Dogs and Humans Based on the CFFROD

The CFFROD can be used to explain the fractional order transmission model for rabies dynamics in a nonlinear fractional differential equation system.

$$^{CF}_{0}D_t^{\Phi} S_1 = \pi_1 - (\mu + \alpha_1 + \gamma_1)S_1 + \lambda_1 R_1$$

$$^{CF}_{0}D_t^{\Phi} E_1 = \alpha_1 S_1 - (\mu + \beta_1)E_1$$

$$^{CF}_{0}D_t^{\Phi} I_1 = \beta_1 E_1 - (\mu + \mu_1 + \delta_1)I_1$$

$$^{CF}_{0}D_t^{\Phi} R_1 = \gamma_1 S_1 - (\lambda_1 + \mu)R_1 \quad (7)$$

$$^{CF}_{0}D_t^{\Phi} S_2 = \pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu + \tau_2 + \eta_2)S_2$$

$$^{CF}_{0}D_t^{\Phi} E_2 = \eta_2 S_2 - (\omega_2 + \mu)E_2$$

$$^{CF}_{0}D_t^{\Phi} I_2 = \omega_2 E_2 - (\mu + \mu_2) I_2$$

$$^{CF}_{0}D_t^{\Phi} R_2 = \tau_2 S_2 - (\mu + \theta_2) R_2$$

The comprehensive FROD mathematical model [Equation (6)] for rabies transmission in dogs and people accurately depicts real-world settings, offering useful predictions to influence decision-making and control efforts. This paradigm enables the assessment of future conditions and the implementation of preventative steps well in advance to avert worst-case outcomes. The mathematical model's solution analysis, which includes a local stability study, is shown below, calculating the reproduction number, establishing the presence of equilibrium points, and confirming the positivity of solutions.

$$R_+^8 = \{\xi \in R^8 : \xi \geq 0\}, \xi(t) = [S_1(t), E_1(t), I_1(t), R_1(t), S_2(t), E_2(t), I_2(t), R_2(t)]^T.$$

Lemma 1. *The proposed model [Equation (7)] has a positive, unique solution $\xi(t)$ that lies in R^8.*

Proof of Lemma 1. It is demonstrated by examining the population model's solution positivity that each component remains confined in the positive quadrant. This tendency is inherent in population models, which use variables to describe non-negative values such as population numbers or concentrations. The vector field directing the system tends to approach the positive orthant R^8, reflecting the inherent restrictions of populations, which are not negative.

$$^{CF}_{0}D_t^{\Phi} S_1 = \pi_1 + \lambda_1 R_1 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} E_1 = \alpha_1 S_1 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} I_1 = \beta_1 E_1 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} R_1 = \gamma_1 S_1 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} S_2 = \pi_2 + \delta_1 I_1 + \theta_2 R_2 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} E_2 = \eta_2 S_2 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} I_2 = \omega_2 E_2 \geq 0$$

$$^{CF}_{0}D_t^{\Phi} R_2 = \tau_2 S_2 \geq 0.$$

3.2.1. Disease-Free Equilibrium

Consider the following DFE for the applying model under the Caputo fractional-order operator: $S_1(t), E_1(t), I_1(t), R_1(t), S_2(t), E_2(t), I_2(t), R_2(t)$. Assuming the right sides of the model are set to zero, the outcome are:

$$\pi_1 - (\mu + \alpha_1 + \gamma_1) S_1 + \lambda_1 R_1 = 0$$

$$\alpha_1 S_1 - (\mu + \beta_1) E_1 = 0$$

$$\beta_1 E_1 - (\mu + \mu_1 + \delta_1) I_1 = 0$$

$$\gamma_1 S_1 - (\lambda_1 + \mu) R_1 = 0$$

$$\pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu + \tau_2 + \eta_2) S_2 = 0$$

$$\eta_2 S_2 - (\omega_2 + \mu) E_2 = 0$$

$$\omega_2 E_2 - (\mu + \mu_2) I_2 = 0$$

$$\tau_2 S_2 - (\mu + \theta_2) R_2 = 0$$

$$E_1^0 = (S_1^0, E_1^0, I_1^0, R_1^0, S_2^0, E_2^0, I_2^0, R_2^0) = (S_1^0, 0,0, R_1^0, S_2^0, 0,0, R_2^0)$$
$$= \left(\frac{\pi_1+\lambda_1 R_1}{(\mu+\alpha_1+\gamma_1)}, 0,0, \frac{\gamma_1 S_1}{(\lambda_1+\mu)}, \frac{\pi_2+\theta_2 R_2}{(\mu+\tau_2+\eta_2)}, 0,0, \frac{\tau_2 S_2}{(\mu+\theta_2)}\right).$$

3.2.2. Disease-Endemic Equilibrium

The endemic equilibrium point in our rabies transmission model is a stable state where the population of infected individuals remains unchanged. We can reach this equilibrium by modifying some of the model parameters to zero, which, over time, shows a constant level of infection in the population. $E = S_1^*, E_1^*, I_1^*, R_1^*, S_2^*, E_2^*, I_2^*, R_2^*$ is what we obtain. Where,

$$S_1^* = \frac{\pi_1\mu + \pi_1\lambda_1 + \lambda_1\gamma_1 S_1}{(\mu + \alpha_1 + \gamma_1)(\lambda_1 + \mu)}$$

$$E_1^* = \frac{\alpha_1(\pi_1 + \lambda_1 R_1)}{(\mu + \alpha_1 + \gamma_1)(\mu + \beta_1)}$$

$$I_1^* = \frac{\beta_1\alpha_1 S_1}{(\mu + \beta_1)(\mu + \mu_1 + \delta_1)}$$

$$R_1^* = \frac{\gamma_1(\pi_1 + \lambda_1 R_1)}{(\mu + \alpha_1 + \gamma_1)(\lambda_1 + \mu)}$$

$$S_2^* = \frac{\pi_2(\mu + \theta_2) + \delta_1 I_1(\mu + \theta_2) + \theta_2\tau_2 S_2}{(\mu + \tau_2 + \eta_2)(\mu + \theta_2)}$$

$$E_2^* = \frac{\pi_2\tau_2(\mu + \theta_2) + \tau_2\delta_1 I_1 + \theta_2\tau_2 R_2}{(\mu + \tau_2 + \eta_2)(\omega_2 + \mu)}$$

$$I_2^* = \frac{\omega_2\eta_2 S_h}{(\omega_2 + \mu)(\mu + \mu_2)}$$

$$R_2^* = \frac{\tau_2(\pi_2 + \delta_1 I_1 + \theta_2 R_2)}{(\mu + \tau_2 + \eta_2)(\mu + \theta_2)}$$

3.2.3. Reproduction Number

The basic reproduction number in epidemiological Modelling, denoted as R_0, can be determined using the next-generation matrix technique. This method requires defining two matrices: matrix F, which illustrates the flow of individuals from one compartment (e.g., susceptible) to another (e.g., infected), and matrix V, which signifies how quickly infected individuals progress to new infections. In the next-generation matrix approach, R_0 is computed as the product of matrices V and F, with F capturing compartmental transitions and V encapsulating infectiousness and contact rates.

$$f_i = \begin{bmatrix} \alpha_1 S_1 \\ 0 \\ \eta_2 S_2 \\ 0 \end{bmatrix}, F = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v_i = \begin{bmatrix} (\mu + \beta_1)E_1 \\ (\mu + \mu_1 + \delta_1)I_1 \\ (\omega_2 + \mu)E_2 \\ (\mu + \mu_2)I_2 \end{bmatrix}, V = \begin{bmatrix} (\mu + \beta_1) & 0 & 0 & 0 \\ 0 & (\mu + \mu_1 + \delta_1) & 0 & 0 \\ 0 & 0 & (\omega_2 + \mu) & 0 \\ 0 & 0 & 0 & (\mu + \mu_2) \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{(\mu+\beta_1)} & 0 & 0 & 0 \\ 0 & \frac{1}{(\mu+\mu_1+\delta_1)I_1} & 0 & 0 \\ 0 & 0 & \frac{1}{(\omega_2+\mu)} & 0 \\ 0 & 0 & 0 & \frac{1}{(\mu+\mu_2)} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\alpha_1}{(\mu+\beta_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\eta_2}{(\omega_2+\mu)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the fundamental reproduction number is,

$$R_0 = FV^{-1} = \frac{\alpha_1(\omega_2+\mu)+\eta_2(\mu+\beta_1)}{(\mu+\beta_1)(\omega_2+\mu)}.$$

4. The CFFROD Mathematical Model Solutions

In this study, we will employ fixed point theory and fractional derivatives to investigate the uniqueness and existence of solutions for the FROD differential equation described in Equation (7).

$$S_1(t) - S_1(0) = {}^{CF}_{0}D_t^{\Phi}\{\pi_1 - (\mu+\alpha_1+\gamma_1)S_1 + \lambda_1 R_1\}$$

$$E_1(t) - E_1(0) = {}^{CF}_{0}D_t^{\Phi}\{\alpha_1 S_1 - (\mu+\beta_1)E_1\}$$

$$I_1(t) - I_1(0) = {}^{CF}_{0}D_t^{\Phi}\{\beta_1 E_1 - (\mu+\mu_1+\delta_1)I_1\}$$

$$R_1(t) - R_1(0) = {}^{CF}_{0}D_t^{\Phi}\{\gamma_1 S_1 - (\lambda_1+\mu)R_1\}$$

$$S_2(t) - S_2(0) = {}^{CF}_{0}D_t^{\Phi}\{\pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu+\tau_2+\eta_2)S_2\}$$

$$E_2(t) - E_2(0) = {}^{CF}_{0}D_t^{\Phi}\{\eta_2 S_2 - (\omega_2+\mu)E_2\}$$

$$I_2(t) - I_2(0) = {}^{CF}_{0}D_t^{\Phi}\{\omega_2 E_2 - (\mu+\mu_2)I_2\}$$

$$R_2(t) - R_2(0) = {}^{CF}_{0}D_t^{\Phi}\{\tau_2 S_2 - (\mu+\theta_2)R_2\}$$

Now using the FROD in Equation (8),

$$S_1(t) - S_1(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\pi_1 - (\mu+\alpha_1+\gamma_1)S_1 + \lambda_1 R_1\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\pi_1 - (\mu+\alpha_1+\gamma_1)S_1 + \lambda_1 R_1\}dy,$$

$$E_1(t) - E_1(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\alpha_1 S_1 - (\mu+\beta_1)E_1\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\alpha_1 S_1 - (\mu+\beta_1)E_1\}dy,$$

$$I_1(t) - I_1(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\beta_1 E_1 - (\mu+\mu_1+\delta_1)I_1\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\beta_1 E_1 - (\mu+\mu_1+\delta_1)I_1\}dy,$$

$$R_1(t) - R_1(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\gamma_1 S_1 - (\lambda_1 + \mu)R_1\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\gamma_1 S_1 - (\lambda_1 + \mu)R_1\}dy, \quad (9)$$

$$S_2(t) - S_2(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu + \tau_2 + \eta_2)S_2\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu + \tau_2 + \eta_2)S_2\}dy,$$

$$E_2(t) - E_2(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\eta_2 S_2 - (\omega_2 + \mu)E_2\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\eta_2 S_2 - (\omega_2 + \mu)E_2\}dy,$$

$$I_2(t) - I_2(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\omega_2 E_2 - (\mu + \mu_2)I_2\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\omega_2 E_2 - (\mu + \mu_2)I_2\}dy,$$

$$R_2(t) - R_2(0) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\{\tau_2 S_2 - (\mu + \theta_2)R_2\} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \{\tau_2 S_2 - (\mu + \theta_2)R_2\}dy.$$

The equation mentioned earlier may be expressed as follows after being simplified using Equation (9),

$$\begin{aligned}
\mathcal{A}_1(t, S_1) &= \pi_1 - (\mu + \alpha_1 + \gamma_1)S_1 + \lambda_1 R_1,\\
\mathcal{A}_2(t, E_1) &= \alpha_1 S_1 - (\mu + \beta_1)E_1,\\
\mathcal{A}_3(t, I_1) &= \beta_1 E_1 - (\mu + \mu_1 + \delta_1)I_1,\\
\mathcal{A}_4(t, R_1) &= \gamma_1 S_1 - (\lambda_1 + \mu)R_1,\\
\mathcal{A}_5(t, S_2) &= \pi_2 + \delta_1 I_1 + \theta_2 R_2 - (\mu + \tau_2 + \eta_2)S_2,\\
\mathcal{A}_6(t, E_2) &= \eta_2 S_2 - (\omega_2 + \mu)E_2,\\
\mathcal{A}_7(t, I_2) &= \omega_2 E_2 - (\mu + \mu_2)I_2,\\
\mathcal{A}_8(t, R_2) &= \tau_2 S_2 - (\mu + \theta_2)R_2.
\end{aligned} \quad (10)$$

Theorem 2. *If the suggested fractional-order mathematical model systems meet the requirements listed below, then the kernels* $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7)$ *and* $\mathcal{A}_8$ *in Equation (7) will adhere to Lipschitz and contraction conditions:* $0 \le (\mu + \alpha_1 + \gamma_1) < 1$.

Proof of Theorem 2. Considering the functions S_1 and $S_{1(1)}$, let's begin with $\mathcal{A}_1$ and then apply the following procedure:

$$\mathcal{A}_1(t, S_1) - \mathcal{A}_1\big(t, S_{1(1)}\big) = -\mu(S_1(t) - S_1(t_1)) - \alpha_1(S_1(t) - S_1(t_1)) - \gamma_1(S_1(t) - S_1(t_1)).$$

Applying the norm to Equation (10) and simplifying, we obtain the result.

$$\begin{aligned}
\big\|\mathcal{A}_1(t, S_1) - \mathcal{A}_1\big(t, S_{1(1)}\big)\big\| &\le \|\mu(S_1(t) - S_1(t_1))\| + \|\alpha_1(S_1(t) - S_1(t_1))\| + \|\gamma_1(S_1(t) - S_1(t_1))\|\\
&\le (\mu + \alpha_1 + \gamma_1)\|(S_1(t) - S_1(t_1))\|
\end{aligned}$$

After simplification, we obtain the following Equation (11),

$$\left\|\mathcal{A}_1(t,S_1)-\mathcal{A}_1\left(t,S_{1(1)}\right)\right\| \leq \chi_1\|(S_1(t)-S_1(t_1))\| \tag{11}$$

Where $\chi_1 = \mu + \alpha_1 + \gamma_1$

Given that $\mathcal{A}_1$ fulfills the Lipschitz condition, as implied by Equation (10), the condition $0 \leq (\mu + \alpha_1 + \gamma_1) < 1$ predicts contraction. In a similar vein, we may determine that the following is the Lipschitz condition for others:

After simplification, we obtain the following Equation (12),

$$\begin{aligned}
&\left\|\mathcal{A}_2(t,E_1)-\mathcal{A}_2\left(t,E_{1(1)}\right)\right\| \leq \chi_2\|(E_1(t)-E_1(t_1))\|,\\
&\left\|\mathcal{A}_3(t,I_1)-\mathcal{A}_3\left(t,I_{1(1)}\right)\right\| \leq \chi_3\|(I_1(t)-I_1(t_1))\|,\\
&\left\|\mathcal{A}_4(t,R_1)-\mathcal{A}_4\left(t,R_{1(1)}\right)\right\| \leq \chi_4\|(R_1(t)-R_1(t_1))\|,\\
&\left\|\mathcal{A}_5(t,S_2)-\mathcal{A}_5\left(t,S_{2(1)}\right)\right\| \leq \chi_5\|(S_2(t)-S_2(t_1))\|,\\
&\left\|\mathcal{A}_6(t,E_2)-\mathcal{A}_6\left(t,E_{2(1)}\right)\right\| \leq \chi_6\|(E_2(t)-E_2(t_1))\|,\\
&\left\|\mathcal{A}_7(t,I_2)-\mathcal{A}_7\left(t,I_{2(1)}\right)\right\| \leq \chi_7\|(I_2(t)-I_2(t_1))\|,\\
&\left\|\mathcal{A}_8(t,R_2)-\mathcal{A}_8\left(t,R_{2(1)}\right)\right\| \leq \chi_8\|(R_2(t)-R_2(t_1))\|.
\end{aligned} \tag{12}$$

Now, Equation (9) can be expressed as:

$$S_1(t)-S_1(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_1(t,S_1)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_1(y,S_1))dy,$$

$$E_1(t)-E_1(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_2(t,E_1)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_2(y,E_1))dy,$$

$$I_1(t)-I_1(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_3(t,I_1)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_3(y,I_1))dy,$$

$$R_1(t)-R_1(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_4(t,R_1)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_4(y,R_1))dy,$$

$$S_2(t)-S_2(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_5(t,S_2)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_5(y,S_2))dy,$$

$$E_2(t)-E_2(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_6(t,E_2)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_6(y,E_2))dy,$$

$$I_2(t)-I_2(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_7(t,I_2)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_7(y,I_2))dy,$$

$$R_2(t)-R_2(0)=\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_8(t,R_2)+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t(\mathcal{A}_8(y,R_2))dy.$$

Now, using the Recursive relation, we obtain,

$$S_{1(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_1(t, S_{1(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_1(y, S_{1(n-1)}))dy,$$

$$E_{1(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_2(t, E_{1(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_2(y, E_{1(n-1)}))dy,$$

$$I_{1(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_3(t, I_{1(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_3(y, I_{1(n-1)}))dy,$$

$$R_{1(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_4(t, R_{1(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_4(y, R_{1(n-1)}))dy,$$

$$\mathrm{S}_{2(\mathrm{n})}(\mathrm{t}) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_5(\mathrm{t}, \mathrm{S}_{2(\mathrm{n}-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^{\mathrm{t}} (\mathcal{A}_5(\mathrm{y}, \mathrm{S}_{2(\mathrm{n}-1)}))\mathrm{dy},$$

$$E_{2(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_6(t, E_{2(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_6(y, E_{2(n-1)}))dy,$$

$$I_{2(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_7(t, I_{2(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_7(y, I_{2(n-1)}))dy,$$

$$R_{2(n)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\mathcal{A}_8(t, R_{2(n-1)}) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t (\mathcal{A}_8(y, R_{2(n-1)}))dy.$$

Using the requirements listed below:

$$S_1^0(t) = S_1(0), E_1^0(t) = E_1(0), I_1^0(t) = I_1(0), R_1^0(t) = R_1(0), S_2^0(t) = S_2(0),$$
$$E_2^0(t) = E_2(0), I_2^0(t) = I_2(0), R_2^0(t) = R_2(0).$$

Furthermore, using the difference of successive terms, we found out as:

$$\mathcal{M}_1 = S_{1(n)}(t) - S_{1(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_1\left(t, S_{1(n-1)}\right) - \mathcal{A}_1\left(t, S_{1(n-2)}\right)\right)$$
$$+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t\left(\mathcal{A}_1\left(t, S_{1(n-1)}\right) - \mathcal{A}_1(t, S_{1(n-2)})\right)dy,$$

$$\mathcal{M}_2 = E_{1(n)}(t) - E_{1(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_2\left(t, E_{1(n-1)}\right) - \mathcal{A}_2\left(t, E_{1(n-2)}\right)\right)$$
$$+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t\left(\mathcal{A}_2\left(t, E_{1(n-1)}\right) - \mathcal{A}_2(t, E_{1(n-2)})\right)dy,$$

$$\mathcal{M}_3 = I_{1(n)}(t) - I_{1(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_3\left(t, I_{1(n-1)}\right) - \mathcal{A}_3\left(t, I_{1(n-2)}\right)\right)$$
$$+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t\left(\mathcal{A}_3\left(t, I_{1(n-1)}\right) - \mathcal{A}_3(t, I_{1(n-2)})\right)dy,$$

$$\mathcal{M}_4 = R_{1(n)}(t) - R_{1(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_4\left(t, R_{1(n-1)}\right) - \mathcal{A}_4\left(t, R_{1(n-2)}\right)\right)$$
$$+\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t\left(\mathcal{A}_4\left(t, R_{1(n-1)}\right) - \mathcal{A}_4(t, R_{1(n-2)})\right)dy,$$

$$\mathcal{M}_5 = S_{2(n)}(t) - S_{2(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_5\left(t, S_{2(n-1)}\right) - \mathcal{A}_5\left(t, S_{2(n-2)}\right)\right) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left(\mathcal{A}_5\left(t, S_{2(n-1)}\right) - \mathcal{A}_5(t, S_{2(n-2)})\right)\mathrm{dy},$$

$$\mathcal{M}_6 = E_{2(n)}(t) - E_{2(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_6\left(t, E_{2(n-1)}\right) - \mathcal{A}_6(t, E_{2(n-2)})\right) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left(\mathcal{A}_6\left(t, E_{2(n-1)}\right) - \mathcal{A}_6(t, E_{2(n-2)})\right)dy,$$

$$\mathcal{M}_7 = I_{2(n)}(t) - I_{2(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_7\left(t, I_{2(n-1)}\right) - \mathcal{A}_7\left(t, I_{2(n-2)}\right)\right) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left(\mathcal{A}_7\left(t, I_{2(n-1)}\right) - \mathcal{A}_7(t, I_{2(n-2)})\right)dy,$$

$$\mathcal{M}_8 = R_{2(n)}(t) - R_{2(n-1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_8\left(t, R_{2(n-1)}\right) - \mathcal{A}_8(t, R_{2(n-2)})\right) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left(\mathcal{A}_8\left(t, R_{2(n-1)}\right) - \mathcal{A}_8(t, R_{2(n-2)})\right)dy.$$

$$S_{1(n)}(t) = \sum_i^n \mathcal{M}_{1i}(t), E_{1(n)}(t) = \sum_i^n \mathcal{M}_{2i}(t), I_{1(n)}(t) = \sum_i^n \mathcal{M}_{3i}(t), R_{1(n)}(t) = \sum_i^n \mathcal{M}_{4i}(t),$$

$$S_{2(n)}(t) = \sum_i^n \mathcal{M}_{5i}(t), E_{2(n)}(t) = \sum_i^n \mathcal{M}_{6i}(t), I_{2(n)}(t) = \sum_i^n \mathcal{M}_{7i}(t), R_{2(n)}(t) = \sum_i^n \mathcal{M}_{8i}(t).$$

Estimating by using the same procedure, we have Equation (13) as follows:

$$\begin{aligned}\|\mathcal{M}_{1n}\| &= \left\|S_{1(n)}(t) - S_{1(n-1)}(t)\right\| \\ &= \left\| \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_1\left(t, S_{1(n-1)}\right) - \mathcal{A}_1(t, S_{1(n-2)})\right) \right. \\ &\quad \left. + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left(\mathcal{A}_1\left(t, S_{1(n-1)}\right) - \mathcal{A}_1(t, S_{1(n-2)})\right)dy \right\|.\end{aligned} \tag{13}$$

By employing the triangle inequality, Equation (13) can be transformed:

$$\left\|S_{1(n)}(t) - S_{1(n-1)}(t)\right\| \leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left\|\left(\mathcal{A}_1\left(t, S_{1(n-1)}\right) - \mathcal{A}_1\left(t, S_{1(n-2)}\right)\right)\right\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\left\|\int_0^t \left(\mathcal{A}_1\left(t, S_{1(n-1)}\right) - \mathcal{A}_1(t, S_{1(n-2)})\right)dy\right\|$$

Kernel satisfying the Lipschitz condition, thus we have:

$$\left\|S_{1(n)}(t) - S_{1(n-1)}(t)\right\| \leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1\|(S_1(t) - S_1(t_1))\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_1\int_0^t \|(S_1(t) - S_1(t_1))\|dy$$

Now, we get:

$$\|\mathcal{M}_{1n}(t)\| \leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1\left\|\mathcal{M}_{1(n-1)}(t)\right\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_1\int_0^t \left\|\mathcal{M}_{1(n-1)}(t)\right\|dy.$$

Similarly,

$$\begin{aligned}
\|\mathcal{M}_{2n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_2\|\mathcal{M}_{2(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_2\int_0^t\|\mathcal{M}_{2(n-1)}(t)\|dy,\\
\|\mathcal{M}_{3n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_3\|\mathcal{M}_{3(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_3\int_0^t\|\mathcal{M}_{3(n-1)}(t)\|dy,\\
\|\mathcal{M}_{4n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_4\|\mathcal{M}_{4(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_4\int_0^t\|\mathcal{M}_{4(n-1)}(t)\|dy,\\
\|\mathcal{M}_{5n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_5\|\mathcal{M}_{5(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_5\int_0^t\|\mathcal{M}_{5(n-1)}(t)\|dy,\\
\|\mathcal{M}_{6n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_6\|\mathcal{M}_{6(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_6\int_0^t\|\mathcal{M}_{6(n-1)}(t)\|dy,\\
\|\mathcal{M}_{7n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_7\|\mathcal{M}_{7(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_7\int_0^t\|\mathcal{M}_{7(n-1)}(t)\|dy,\\
\|\mathcal{M}_{8n}(t)\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_8\|\mathcal{M}_{8(n-1)}(t)\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_8\int_0^t\|\mathcal{M}_{8(n-1)}(t)\|dy.
\end{aligned} \tag{14}$$

The outcomes given in Equation (14) is used to show that the solution has an existence.

Theorem 3. *If the following requirements are satisfied, the suggested fractional order mathematical model system provides precise coupled solutions for Rabies transmission. That is, we can discover* t_0 *so that* $\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1 + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_1 t_0 < 1$

Where $S_1(t), E_1(t), I_1(t), R_1(t), S_2(t), E_2(t), I_2(t)$ and $R_2(t)$ are bounded functions. Therefore, Kernels satisfy the Lipschitz condition:

$$\begin{aligned}
\|\mathcal{M}_{1n}(t)\| &\leq \|S_{1n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_1 t\right)\right]^n,\\
\|\mathcal{M}_{2n}(t)\| &\leq \|E_{1n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_2\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_2 t\right)\right]^n,\\
\|\mathcal{M}_{3n}(t)\| &\leq \|I_{1n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_3\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_3 t\right)\right]^n,\\
\|\mathcal{M}_{4n}(t)\| &\leq \|R_{1n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_4\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_4 t\right)\right]^n,\\
\|\mathcal{M}_{5n}(t)\| &\leq \|S_{2n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_5\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_5 t\right)\right]^n,\\
\|\mathcal{M}_{6n}(t)\| &\leq \|E_{2n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_6\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_6 t\right)\right]^n,\\
\|\mathcal{M}_{7n}(t)\| &\leq \|I_{2n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_7\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_7 t\right)\right]^n,\\
\|\mathcal{M}_{8n}(t)\| &\leq \|R_{2n}(0)\|\left[\left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_8\right) + \left(\frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_8 t\right)\right]^n.
\end{aligned}$$

Now, follow the following procedure:

$$S_1(t) - S_1(0) = S_{1n}(t) - \Xi_{1n}(t),$$
$$E_1(t) - E_1(0) = E_{1n}(t) - \Xi_{2n}(t),$$
$$I_1(t) - I_1(0) = I_{1n}(t) - \Xi_{3n}(t),$$
$$R_1(t) - R_1(0) = R_{1n}(t) - \Xi_{4n}(t),$$
$$S_2(t) - S_2(0) = S_{2n}(t) - \Xi_{5n}(t),$$
$$E_2(t) - E_2(0) = E_{2n}(t) - \Xi_{6n}(t),$$
$$I_2(t) - I_2(0) = I_{2n}(t) - \Xi_{7n}(t),$$
$$R_2(t) - R_2(0) = R_{2n}(t) - \Xi_{8n}(t).$$

Therefore, we have:

$$\|\Xi_{1n}(t)\| = \left\| \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)} \left(\mathcal{A}_1\left(t, S_{1(n)}\right) - \mathcal{A}_1(t, S_{1(n-1)})\right) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)} \int_0^t \left(\mathcal{A}_1\left(t, S_{1(n)}\right) - \mathcal{A}_1(t, S_{1(n-1)})\right) dy \right\|$$

$$\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)} \left\| \left(\mathcal{A}_1\left(t, S_{1(n)}\right) - \mathcal{A}_1\left(t, S_{1(n-1)}\right)\right) \right\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)} \int_0^t \left\| \left(\mathcal{A}_1\left(t, S_{1(n)}\right) - \mathcal{A}_1(t, S_{1(n-1)})\right) dy \right\|,$$

$$\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)} \chi_1 \left\| (S_{1(n)} - S_{1(n-1)}) \right\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)} \chi_1 \left\| (S_{1(n)} - S_{1(n-1)}) \right\| t.$$

We followed,

$$\|\Xi_{1n}(t)\| \leq \left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)} t \right)^{n+1} \chi_1^{n+1} c.$$

At t_0,

$$\|\Xi_{1n}(t)\| \leq \left(\frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)} + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)} t_0 \right)^{n+1} \chi_1^{n+1} c. \tag{15}$$

We can deduce the following from Equation (15):

$$\|\Xi_{1n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty.$$

In the same way, we have

$$\|\Xi_{2n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty,$$
$$\|\Xi_{3n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty,$$
$$\|\Xi_{4n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty,$$
$$\|\Xi_{5n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty,$$
$$\|\Xi_{6n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty,$$
$$\|\Xi_{7n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty,$$
$$\|\Xi_{8n}(t)\| \longrightarrow 0, \qquad n \longrightarrow \infty.$$

This indicates that there is a solution to the proposed fractional-order mathematical system [Equation (7)]. If we assume that $S_{1(1)}(t), E_{1(1)}(t), I_{1(1)}(t), R_{1(1)}(t), S_{2(1)}(t), E_{2(1)}((t), I_{2(1)}(t)$ and $R_{2(1)}(t)$, is another solution to system [Equation (7)], then,

$$S_1(t) - S_{1(1)}(t) = \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left(\mathcal{A}_1(t, S_1) - \mathcal{A}_1\left(t, S_{1(1)}\right)\right) + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left(\mathcal{A}_1(t, S_1) - \mathcal{A}_1\left(t, S_{1(1)}\right)\right) dy. \tag{16}$$

With the support of norm, Equation (16) takes the form:

$$\left\|S_1(t) - S_{1(1)}(t)\right\| \leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\left\|\left(\mathcal{A}_1(t, S_1) - \mathcal{A}_1\left(t, S_{1(1)}\right)\right)\right\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \left\|\left(\mathcal{A}_1(t, S_1) - \mathcal{A}_1\left(t, S_{1(1)}\right)\right)\right\| dy,$$

Furthermore, the Lipschitz condition of the kernel yields:

$$\begin{aligned}\left\|S_1(t) - S_{1(1)}(t)\right\| &\leq \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1\left\|(S_1(t) - S_{1(1)}(t))\right\| + \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\int_0^t \chi_1\left\|(S_1(t) - S_{1(1)}(t))\right\| dy, \\ \left\|S_1(t) - S_{1(1)}(t)\right\|&\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_1 t\right) \leq 0.\end{aligned} \tag{17}$$

Therefore we have:

$$\left\|E_1(t) - E_{1(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_2 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_2 t\right) \leq 0,$$

$$\left\|I_1(t) - I_{1(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_3 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_3 t\right) \leq 0,$$

$$\left\|R_1(t) - R_{1(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_4 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_4 t\right) \leq 0,$$

$$\left\|S_2(t) - S_{2(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_5 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_5 t\right) \leq 0,$$

$$\left\|E_2(t) - E_{2(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_6 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_6 t\right) \leq 0,$$

$$\left\|I_2(t) - I_{2(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_7 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_7 t\right) \leq 0,$$

$$\left\|R_2(t) - R_{2(1)}(t)\right\|\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_8 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_8 t\right) \leq 0.$$

Theorem 4. *A unique solution is provided by the proposed fraction order mathematical model system [Equation (7)], if*

$$\left(1 - \frac{2(1-\Phi)}{(2-\Phi)\Psi(\Phi)}\chi_1 - \frac{2\Phi}{(2-\Phi)\Psi(\Phi)}\chi_1 t\right) > 0. \tag{18}$$

Proof of Theorem 4: If Equation (18) is true, Equation (17) indicates that $\|S_1(t) - S_{1(1)}(t)\| = 0$. So, we get:

$$S_1(t) = S_{1(1)}(t)$$

Furthermore, we have:

$$E_1(t) = E_{1(1)}(t),$$
$$I_1(t) = I_{1(1)}(t),$$
$$R_1(t) = R_{1(1)}(t),$$
$$S_2(t) = S_{2(1)}(t),$$
$$E_2(t) = E_{2(1)}(t),$$
$$I_2(t) = I_{2(1)}(t),$$
$$R_2(t) = R_{2(1)}(t).$$

As a result, the mathematical model [Equation (7)] for fractional order proposed is unique.

5. Numerical Scheme

We have discussed a new numerical technique for discretizing fractional differential equations in this inquiry part. It was suggested by Agangana and Owolabi and makes use of the CFFROD. This method was used by Agangana and Owolabi to solve a particular fractional differential equation.

$${}^{CF}_{0}D^{\Phi}_{t}Z(t) = g\big(t, Z(t)\big), \qquad \text{or}$$

$$\big(t, Z(t)\big) = \frac{\Psi(\Phi)}{(1-\Phi)}\int_0^t Z'(\varrho)exp\left[-\Phi\frac{\Phi}{1-\Phi}(t-\varrho)\right]d\varrho.$$

Given the equation discussed previously and following the fundamental theorem of analysis, we obtain:

$$Z(t) - Z(0) = \frac{(1-\Phi)}{\Psi(\Phi)}g\big(t, Z(t)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^t g\big(\varrho, Z(\varrho)\big)d\varrho.$$

As stated above,

$$Z(t_{n+1}) - Z(0) = \frac{(1-\Phi)}{\Psi(\Phi)}g\big(t_n, Z(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} g\big(t, Z(t)\big)dt, \tag{19}$$

$$Z(t_n) - Z(0) = \frac{(1-\Phi)}{\Psi(\Phi)}g\big(t_{n-1}, Z(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} g\big(t, Z(t)\big)dt. \tag{20}$$

The system of equations below is generated using Equations (19) and (20).

$$Z(t_{n+1}) - Z(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{g\big(t_n, Z(t_n)\big) - g\big(t_{n-1}, Z(t_{n-1})\big)\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} g\big(t, Z(t)\big)dt. \tag{21}$$

Where,

$$\int_{t_n}^{t_{n+1}} g\big(t, Z(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{g(t_n, Z_n)}{h}(t - t_{n-1}) - \frac{g(t_{n-1}, Z_{n-1})}{h}(t - t_n)\right\} dt,$$

$$\int_{t_n}^{t_{n+1}} g\big(t, Z(t)\big)dt = \frac{3h}{2} g(t_n, Z_n) - \frac{h}{2} g(t_{n-1}, Z_{n-1}).$$

As a result of Equation (21), the following expression is obtained.

$$Z(t_{n+1}) - Z(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{g\big(t_n, Z(t_n)\big) - g\big(t_{n-1}, Z(t_{n-1})\big)\} + \frac{3h\Phi}{2\Psi(\Phi)} g(t_n, Z_n) - \frac{h\Phi}{2\Psi(\Phi)} g(t_{n-1}, Z_{n-1}).$$

$$Z(t_{n+1}) - Z(t_n) = \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right) g(t_n, Z_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right) g(t_{n-1}, Z_{n-1}). \quad (22)$$

$$Z(t_{n+1}) = Z(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right) g(t_n, Z_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right) g(t_{n-1}, Z_{n-1}).$$

The comparable two-step Adamas-Bashforth numerical method for the CFFROD is given by Equation (22).

Theorem 5. *Assuming that g is a continuous bounded function for the CFFROD and z (t) is the solution to the fractional order differential equation* ${}_0^{CF}D_t^{\Phi} Z(t) = g\big(t, Z(t)\big)$*, then*

$$Z(t_{n+1}) = Z(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right) g(t_n, Z_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right) g(t_{n-1}, Z_{n-1}) + \varsigma_{\Phi}^{n},$$

Where $\|\varsigma_{\Phi}^{n}\| \leq \mathbb{K}$.

6. The Fractional Order Mathematical Model of Rabies Transmission and its Numerical Approach

In this research, we simulate the novel CFFROD for fractional order Rabies illness within the given model system [Equation (7)] using a recently constructed numerical technique. We first use the calculus basic theorem to rearrange the model system [Equation (7)] into the following fractional equation in order to estimate the solution of this model system using numerical iteration with this approach.

$$S_1(t) - S_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)} \mathcal{A}_1\big(t, S_1(t)\big) + \frac{\Phi}{\Psi(\Phi)} \int_0^t \mathcal{A}_1\big(\varrho, S_1(\varrho)\big)d\varrho,$$

$$E_1(t) - E_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)} \mathcal{A}_2\big(t, E_1(t)\big) + \frac{\Phi}{\Psi(\Phi)} \int_0^t \mathcal{A}_2\big(\varrho, E_1(\varrho)\big)d\varrho,$$

$$I_1(t) - I_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)} \mathcal{A}_3\big(t, I_1(t)\big) + \frac{\Phi}{\Psi(\Phi)} \int_0^t \mathcal{A}_3\big(\varrho, I_1(\varrho)\big)d\varrho,$$

$$R_1(t) - R_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)} \mathcal{A}_4\big(t, R_1(t)\big) + \frac{\Phi}{\Psi(\Phi)} \int_0^t \mathcal{A}_4\big(\varrho, R_1(\varrho)\big)d\varrho,$$

$$S_2(t) - S_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_5\big(t, S_2(t)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^t \mathcal{A}_5\big(\varrho, S_2(\varrho)\big)d\varrho,$$

$$E_2(t) - E_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_6\big(t, E_2(t)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^t \mathcal{A}_6\big(\varrho, E_2(\varrho)\big)d\varrho,$$

$$I_2(t) - I_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_7\big(t, I_2(t)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^t \mathcal{A}_7\big(\varrho, I_2(\varrho)\big)d\varrho,$$

$$R_2(t) - R_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_8\big(t, R_2(t)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^t \mathcal{A}_8\big(\varrho, R_2(\varrho)\big)d\varrho.$$

Therefore, after simplification, we obtain the following Equation (23),

$$\begin{aligned}
S_1(t_{n+1}) - S_1(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_1\big(t_n, S_1(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_1\big(t, S_1(t)\big)dt,\\
E_1(t_{n+1}) - E_1(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_2\big(t_n, E_1(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_2\big(t, E_1(t)\big)dt,\\
I_1(t_{n+1}) - I_1(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_3\big(t_n, I_1(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_3\big(t, I_1(t)\big)dt,\\
R_1(t_{n+1}) - R_1(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_4\big(t_n, R_1(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_4\big(t, R_1(t)\big)dt,\\
S_2(t_{n+1}) - S_2(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_5\big(t_n, S_2(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_5\big(t, S_2(t)\big)dt,\\
E_2(t_{n+1}) - E_2(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_6\big(t_n, E_2(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_6\big(t, E_2(t)\big)dt,\\
I_2(t_{n+1}) - I_2(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_7\big(t_n, I_2(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_7\big(t, I_2(t)\big)dt,\\
R_2(t_{n+1}) - R_2(0) &= \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_8\big(t_n, R_2(t_n)\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_{n+1}} \mathcal{A}_8\big(t, R_2(t)\big)dt,
\end{aligned} \tag{23}$$

after simplification, we obtain the following Equation (24),

$$S_1(t_n) - S_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_1\big(t_{n-1}, S_1(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_1\big(t, S_1(t)\big)dt,$$

$$E_1(t_n) - E_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_2\big(t_{n-1}, E_1(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_2\big(t, E_1(t)\big)dt,$$

$$I_1(t_n) - I_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_3\big(t_{n-1}, I_1(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_3\big(t, I_1(t)\big)dt,$$

$$R_1(t_n) - R_1(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_4\big(t_{n-1}, R_1(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_4\big(t, R_1(t)\big)dt, \tag{24}$$

$$S_2(t_n) - S_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_5\big(t_{n-1}, S_2(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_5\big(t, S_2(t)\big)dt,$$

$$E_2(t_n) - E_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_6\big(t_{n-1}, E_2(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_6\big(t, E_2(t)\big)dt,$$

$$I_2(t_n) - I_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_7\big(t_{n-1}, I_2(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_7\big(t, I_2(t)\big)dt,$$

$$R_2(t_n) - R_2(0) = \frac{(1-\Phi)}{\Psi(\Phi)}\mathcal{A}_8\big(t_{n-1}, R_2(t_{n-1})\big) + \frac{\Phi}{\Psi(\Phi)}\int_0^{t_n} \mathcal{A}_8\big(t, R_2(t)\big)dt$$

$$S_1(t_{n+1}) - S_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_1\big(t_n, S_1(t_n)\big) - \mathcal{A}_1\big(t_{n-1}, S_1(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_1\big(t, S_1(t)\big)dt,$$

$$E_1(t_{n+1}) - E_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_2\big(t_n, E_1(t_n)\big) - \mathcal{A}_2\big(t_{n-1}, E_1(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_2\big(t, E_1(t)\big)dt,$$

$$I_1(t_{n+1}) - I_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_3\big(t_n, I_1(t_n)\big) - \mathcal{A}_3\big(t_{n-1}, I_1(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_3\big(t, I_1(t)\big)dt,$$

$$R_1(t_{n+1}) - R_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_4\big(t_n, R_1(t_n)\big) - \mathcal{A}_4\big(t_{n-1}, R_1(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_4\big(t, R_1(t)\big)dt,$$

$$S_2(t_{n+1}) - S_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_5\big(t_n, S_1(t_n)\big) - \mathcal{A}_5\big(t_{n-1}, S_2(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_5\big(t, S_2(t)\big)dt,$$

$$E_2(t_{n+1}) - E_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_6\big(t_n, E_2(t_n)\big) - \mathcal{A}_6\big(t_{n-1}, E_2(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_6\big(t, E_2(t)\big)dt,$$

$$I_2(t_{n+1}) - I_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_7\big(t_n, I_2(t_n)\big) - \mathcal{A}_7\big(t_{n-1}, I_2(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_7\big(t, I_2(t)\big)dt,$$

$$R_2(t_{n+1}) - R_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\big\{\mathcal{A}_8\big(t_n, R_2(t_n)\big) - \mathcal{A}_8\big(t_{n-1}, R_2(t_{n-1})\big)\big\} + \frac{\Phi}{\Psi(\Phi)}\int_{t_n}^{t_{n+1}} \mathcal{A}_8\big(t, R_2(t)\big)dt.$$

Where,

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_1\big(t, S_1(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_1(t_n, S_{1n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_1\big(t_{n-1}, S_{1(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_1(t_n, S_{1n}) - \frac{h}{2}\mathcal{A}_1\big(t_{n-1}, S_{1(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_2\big(t, E_1(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_2(t_n, E_{1n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_2\big(t_{n-1}, E_{1(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_2(t_n, E_{1n}) - \frac{h}{2}\mathcal{A}_2\big(t_{n-1}, E_{1(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_3\big(t, I_1(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_3(t_n, I_{1n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_3\big(t_{n-1}, I_{1(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_3(t_n, I_{1n}) - \frac{h}{2}\mathcal{A}_3\big(t_{n-1}, I_{1(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_4\big(t, R_1(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_4(t_n, R_{1n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_4\big(t_{n-1}, R_{1(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_4(t_n, R_{1n}) - \frac{h}{2}\mathcal{A}_4\big(t_{n-1}, R_{1(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_5\big(t, S_2(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_5(t_n, S_{2n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_5\big(t_{n-1}, S_{2(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_5(t_n, S_{2n}) - \frac{h}{2}\mathcal{A}_5\big(t_{n-1}, S_{2(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_6\big(t, E_2(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_6(t_n, E_{2n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_6\big(t_{n-1}, E_{2(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_6(t_n, E_{2n}) - \frac{h}{2}\mathcal{A}_6\big(t_{n-1}, E_{2(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_7\big(t, I_2(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_7(t_n, I_{2n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_7\big(t_{n-1}, I_{2(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_7(t_n, I_{2n}) - \frac{h}{2}\mathcal{A}_7\big(t_{n-1}, I_{2(n-1)}\big).$$

$$\int_{t_n}^{t_{n+1}} \mathcal{A}_8\big(t, R_2(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{\frac{\mathcal{A}_8(t_n, R_{2n})}{h}(t - t_{n-1}) - \frac{\mathcal{A}_8\big(t_{n-1}, R_{2(n-1)}\big)}{h}(t - t_n)\right\}dt,$$

$$= \frac{3h}{2}\mathcal{A}_8(t_n, R_{2n}) - \frac{h}{2}\mathcal{A}_8\big(t_{n-1}, R_{2(n-1)}\big).$$

$$S_1(t_{n+1}) - S_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_1\big(t_n, S_1(t_n)\big) - \mathcal{A}_1\big(t_{n-1}, S_1(t_{n-1})\big)\}$$

$$+ \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_1(t_n, S_{1n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_1\big(t_{n-1}, S_{1(n-1)}\big).$$

$$E_1(t_{n+1}) - E_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_2(t_n, E_1(t_n)) - \mathcal{A}_2(t_{n-1}, E_1(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_2(t_n, E_{1n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_2(t_{n-1}, E_{1(n-1)}).$$

$$I_1(t_{n+1}) - I_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_3(t_n, I_1(t_n)) - \mathcal{A}_3(t_{n-1}, I_1(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_3(t_n, I_{1n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_3(t_{n-1}, I_{1(n-1)}).$$

$$R_1(t_{n+1}) - R_1(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_4(t_n, R_1(t_n)) - \mathcal{A}_4(t_{n-1}, R_1(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_4(t_n, R_{1n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_4(t_{n-1}, R_{1(n-1)}). \quad (25)$$

$$S_2(t_{n+1}) - S_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_5(t_n, S_2(t_n)) - \mathcal{A}_5(t_{n-1}, S_2(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_5(t_n, S_{2n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_5(t_{n-1}, S_{2(n-1)}).$$

$$E_2(t_{n+1}) - E_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_6(t_n, E_2(t_n)) - \mathcal{A}_6(t_{n-1}, E_2(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_6(t_n, E_{2n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_6(t_{n-1}, E_{2(n-1)}).$$

$$I_2(t_{n+1}) - I_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_7(t_n, I_2(t_n)) - \mathcal{A}_7(t_{n-1}, I_2(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_7(t_n, I_{2n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_7(t_{n-1}, I_{2(n-1)}).$$

$$R_2(t_{n+1}) - R_2(t_n) = \frac{(1-\Phi)}{\Psi(\Phi)}\{\mathcal{A}_8(t_n, R_2(t_n)) - \mathcal{A}_8(t_{n-1}, R_2(t_{n-1}))\} + \frac{3h\Phi}{2\Psi(\Phi)}\mathcal{A}_8(t_n, R_{2n}) - \frac{h\Phi}{2\Psi(\Phi)}\mathcal{A}_8(t_{n-1}, R_{2(n-1)}).$$

From Equation (25):

$$S_1(t_{n+1}) = S_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_1(t_n, S_1(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_1(t_{n-1}, S_{1(n-1)}),$$

$$E_1(t_{n+1}) = E_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_2(t_n, E_1(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_2(t_{n-1}, E_{1(n-1)}),$$

$$I_1(t_{n+1}) = I_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_3(t_n, I_1(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_3(t_{n-1}, I_{1(n-1)}),$$

$$R_1(t_{n+1}) = R_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_4(t_n, R_1(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_4(t_{n-1}, R_{1(n-1)}),$$

$$S_2(t_{n+1}) = S_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_5(t_n, S_2(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_5(t_{n-1}, S_{2(n-1)}),$$

$$E_2(t_{n+1}) = E_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_6(t_n, E_2(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_6(t_{n-1}, E_{2(n-1)}),$$

$$I_2(t_{n+1}) = I_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_7(t_n, I_2(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_7\left(t_{n-1}, I_{2(n-1)}\right),$$

$$R_2(t_{n+1}) = R_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_8(t_n, R_2(t_n)) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_8\left(t_{n-1}, R_{2(n-1)}\right).$$

Applying Theorem 5, we get at

$$S_1(t_{n+1}) = S_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_1(t_n, S_{1n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_1\left(t_{n-1}, S_{1(n-1)}\right) + {}^1\varsigma_\Phi^n,$$

$$E_1(t_{n+1}) = E_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_2(t_n, E_{1n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_2\left(t_{n-1}, E_{1(n-1)}\right) + {}^2\varsigma_\Phi^n,$$

$$I_1(t_{n+1}) = I_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_3(t_n, I_{1n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_3\left(t_{n-1}, 1_{1(n-1)}\right) + {}^3\varsigma_\Phi^n,$$

$$R_1(t_{n+1}) = R_1(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_4(t_n, R_{1n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_4\left(t_{n-1}, R_{1(n-1)}\right) + {}^4\varsigma_\Phi^n,$$

$$S_2(t_{n+1}) = S_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_5(t_n, S_{2n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_5\left(t_{n-1}, S_{2(n-1)}\right) + {}^5\varsigma_\Phi^n,$$

$$E_2(t_{n+1}) = E_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_6(t_n, E_{2n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_6\left(t_{n-1}, E_{2(n-1)}\right) + {}^6\varsigma_\Phi^n,$$

$$I_2(t_{n+1}) = I_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_7(t_n, I_{2n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_7\left(t_{n-1}, 1_{2(n-1)}\right) + {}^7\varsigma_\Phi^n,$$

$$R_2(t_{n+1}) = R_2(t_n) + \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{3h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_8(t_n, R_{2n}) - \left(\frac{(1-\Phi)}{\Psi(\Phi)} + \frac{h\Phi}{2\Psi(\Phi)}\right)\mathcal{A}_8\left(t_{n-1}, R_{2(n-1)}\right) + {}^8\varsigma_\Phi^n.$$

Where $\left\| {}^i\varsigma_\Phi^n \right\|_\infty < \frac{\Phi}{\Psi(\Phi)}(n-1)!\, h^{n+1}\mathbb{K}, \quad i = 1,2,\dots,8.$

7. Numerical Simulation

We performed a numerical simulation of the system [Equation (6)] using the MATLAB solver and parameter settings to determine the impact of the general public on the management of the rabies virus. The following values are considered for the selected parameters to perform a numerical simulation and graphical representation of the susceptible, exposed, infected, and recovered rabid virus in the environment for dogs and humans:

Results

This section presents the results of numerical simulations for the proposed fractal fractional-order rabies transmission model, including the use of the Adams–Bashforth method. The effects of changing these key fractional parameters on various population compartments will be shown in **Figures 1–11**. All simulations assume biologically realistic initial values of the model and parameter values.

Figure 1 plots the time-dependent dynamics of all population compartments (susceptible, exposed, infected, and recovered) for dogs and humans with a constant fractional order α_1. The dynamics for the dog populations (S_1, E_1, I_1, R_1) tend to stabilize over time. The human compartments (S_2, E_2, I_2, R_2)also stabilize, which indicates that the model behaves correctly under these conditions.

Figure 2 examines the exposed population of dogs by looking at E_1 for $[5.53,\ 3.28,\ 0.9]$, the different values of the fractional order parameter α_1. The exposed population of dogs grows faster (stabilized sooner) with larger values of the fractional order. This demonstrates the memory of fractional derivative processes in disease infection dynamics.

Figure 3 depicts the infected population of dogs I_1 over time, represented with three different values of the transmission rate $\beta_1 = [0.75,\ 0.045,\ 0.25]$. As we expected, a larger β_1 had a larger infected population, confirming the diseased population is sensitive to rates of contact.

Figure 4 examines the effect of variation in the dog-to-human transmission rate $\delta_1 = [0.05\ ,\ 0.53\ ,\ 1.5]$ on the susceptible human population S_2. The higher the transmission rate, the smaller the number of susceptible individuals in the susceptible class, as more people move from the susceptible class into the exposed and infected classes.

Figure 5 shows the recovered dog population R_1 against different values of recovery rate $\gamma_1 = [0.03,\ 0.45,\ 0.76]$. Higher recovery rates produce greater accumulation of recovered dogs, showing interventions for recovery are effective.

Figure 6 shows the effect of dog vaccination rate $\lambda_1 = [0.05\ ,\ 0.5\ ,\ 3.5]$ on the susceptible dog population S_1. As vaccination rates increase, the susceptible dog population reaches higher levels, with less infection pressure on the susceptible dog population.

Figure 7 shows the exposed human population E_2 for different rates of exposed dog-to-human progression $\eta_2 = [0.623\ ,\ 0.751\ ,\ 0.862]$. The higher η_2 the fewer exposed humans, which is a sign of better control or contact tracing protocols.

Figure 8 illustrates the population of recovered humans $R_2$for different values of the human recovery rate $\tau_2 = [0.832,\ 0.921,\ 0.753]$. Higher human recovery rates can lead to quicker accumulation of recovered infected individuals.

Figure 9 illustrates how the number of human susceptible S_2 changes over time for different values of quarantine efficiency $\theta_2 = [0.832, 0.921\ , 0.753]$. A higher θ_2 value, the better the control to keep individuals in the susceptible class.

Figure 10 shows a bar graph of the total rabies-infected population versus time. The population goes from quick initial increases to saturation over time, demonstrating the long-term boundedness of the model.

Figure 11 shows a 3D surface plot of rabies over time and index, showing a visual summary of the ebbs and flows of the total rabies population across different stages of the simulation.

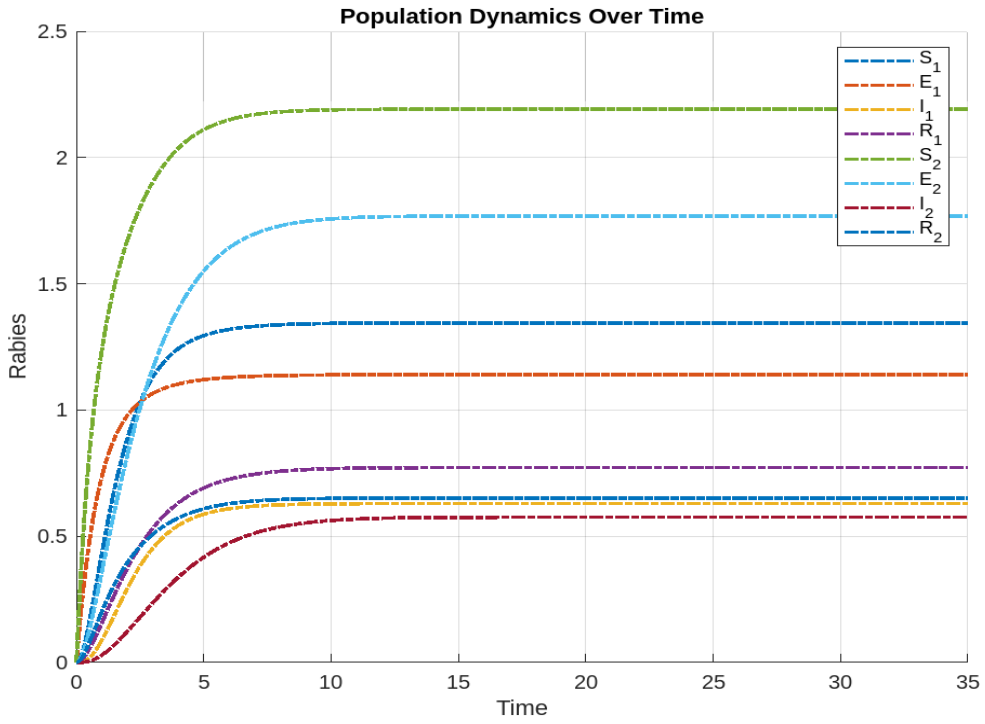

Figure 1. Time-dependent variation in population density for constant fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 0.9, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.521, \pi_2 = 232432, S_2 = 132432, E_2 = 50000, I_2 = 40000, R_2 = 10000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

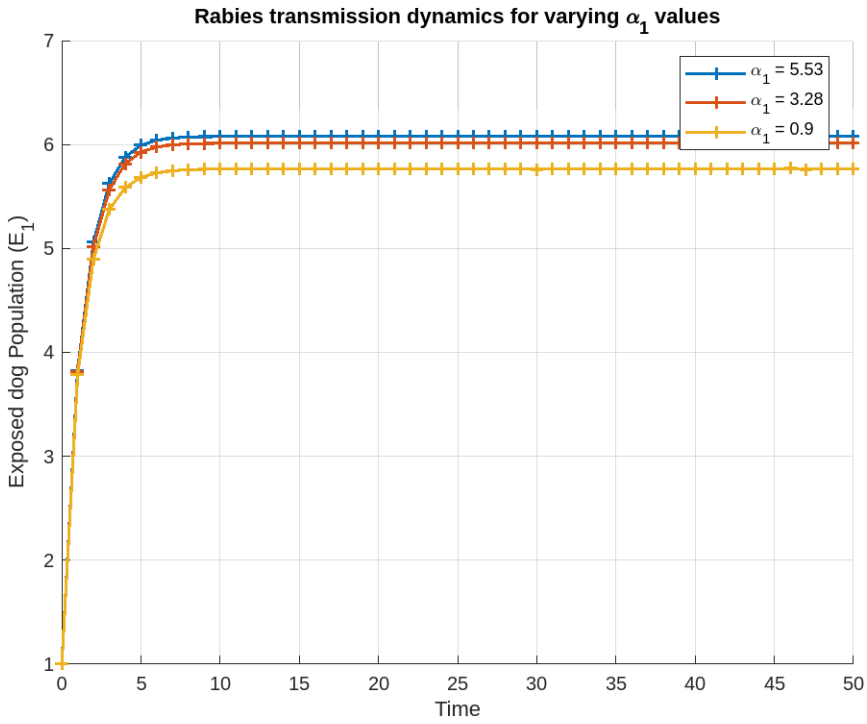

Figure 2. Population variation exposed (dog) over time for a range of fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = [5.53, 3.28, 0.9], \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.521, \pi_2 = 232432, S_2 = 132432, E_2 = 50000, I_2 = 40000, R_2 = 10000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

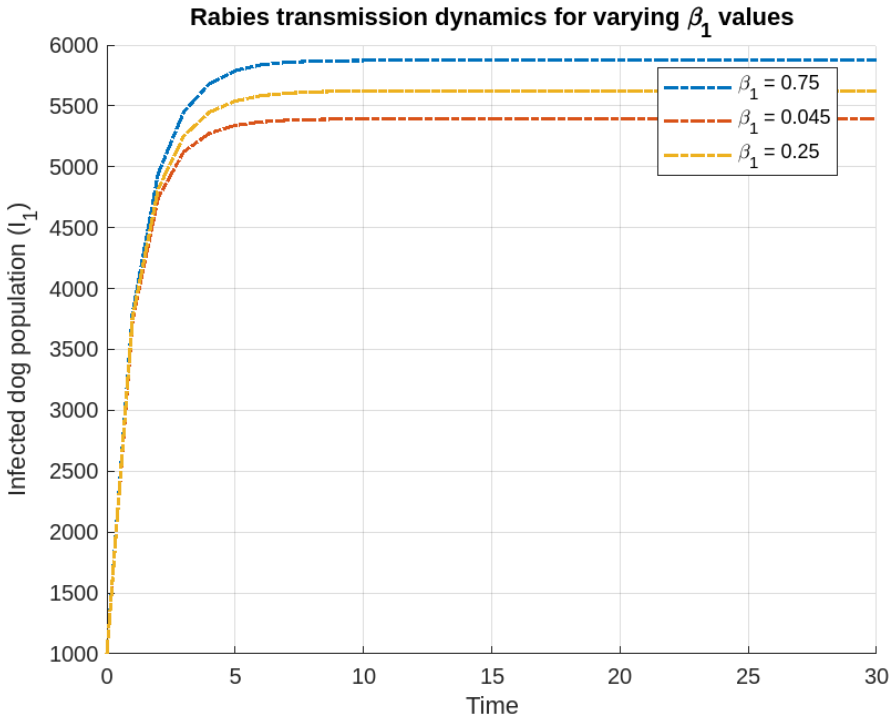

Figure 3. Population variation infected (dog) over time for fractional order parameters.

$\pi_1 = 12532, S_1 = 5000, E_1 = 3000, I_1 = 2532, R_1 = 2000, \alpha_1 = 1.5, \beta_1 = [0.75, 0.045, 0.25], \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.521, \pi_2 = 23243, S_2 = 13243, E_2 = 5000, I_2 = 4000, R_2 = 1000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

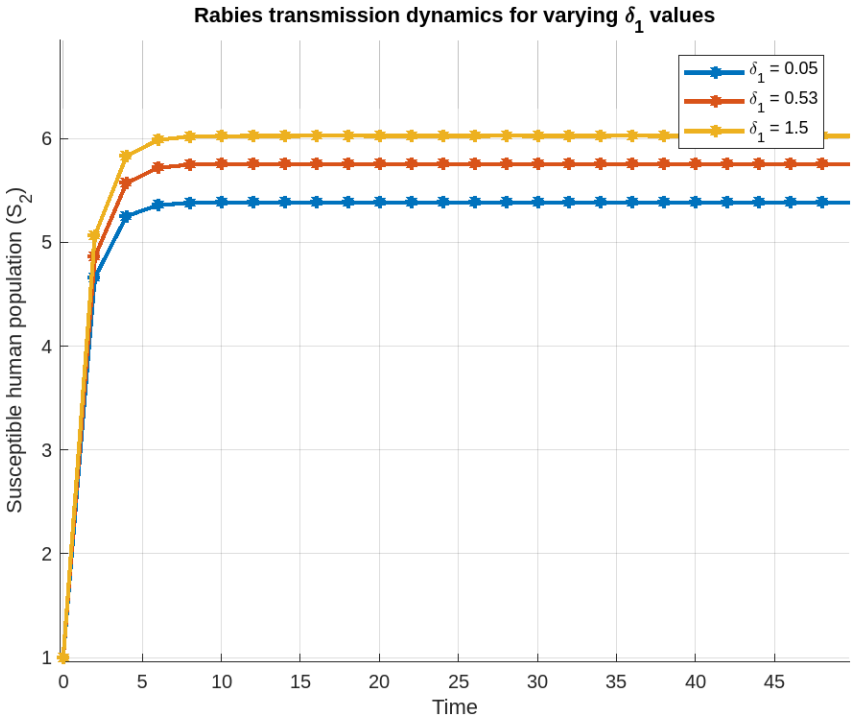

Figure 4. Population variation susceptible (human) over time for fractional order parameters.

$\pi_1 = 12532, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 1.5, \delta_1 = [0.05, 0.53, 1.5], \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25\ \mu = 0.521,\ \pi_2 = 232432, S_2 = 132432, \beta_1 = 0.75,\ \eta_2 = 0.623, E_2 = 50000, I_2 = 40000, R_2 = 10000, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

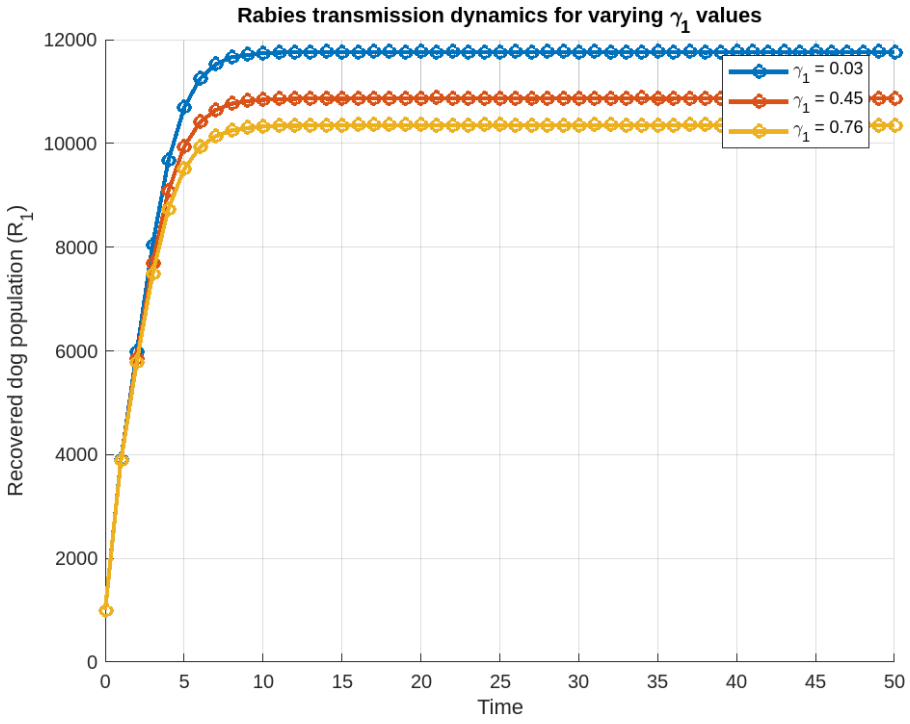

Figure 5. Population variation recovered (dog) over time for fractional order parameters.

$\pi_1 = 12532, S_1 = 5000, E_1 = 3000, I_1 = 2532, R_1 = 2000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = [0.03, 0.45, 0.76],\ \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.521, \pi_2 = 23243, S_2 = 13243, E_2 = 5000, I_2 = 4000, R_2 = 1000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

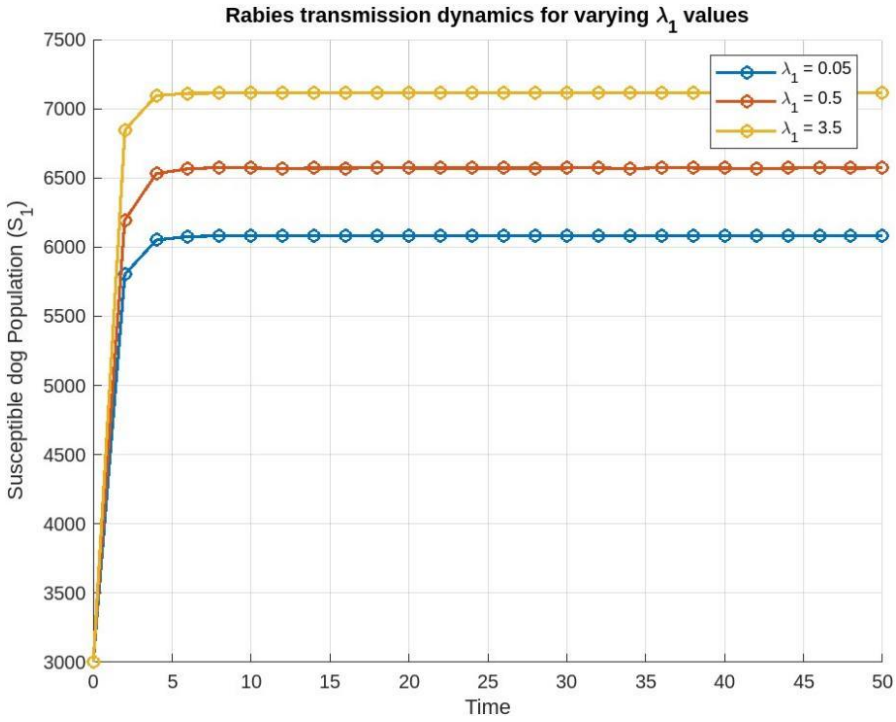

Figure 6. Population variation susceptible (dog) over time for fractional order parameters.

$\pi_1 = 12532, S_1 = 5000, E_1 = 3000, I_1 = 2532, R_1 = 2000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = [0.05, 0.5, 3.5],\ \mu_1 = 0.25, \mu = 0.521, \pi_2 = 23243, S_2 = 13243, E_2 = 5000, I_2 = 4000, R_2 = 1000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

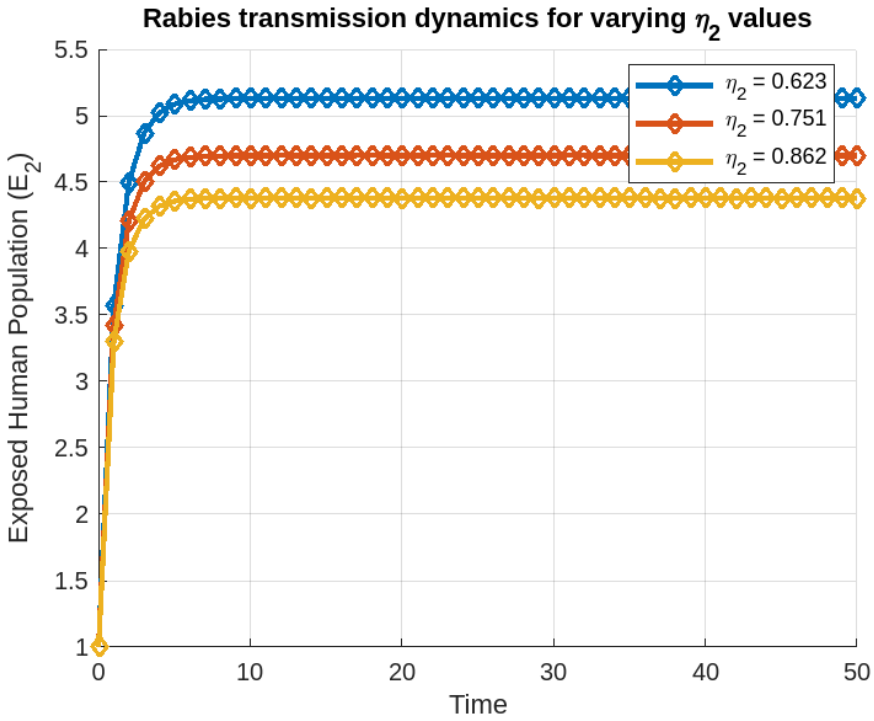

Figure 7. Population variation exposed (human) over time for a range of fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.4, \pi_2 = 232432, S_2 = 132432, E_2 = 50000, I_2 = 40000, R_2 = 10000, \eta_2 = [0.623, 0.751, 0.862], \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.932.$

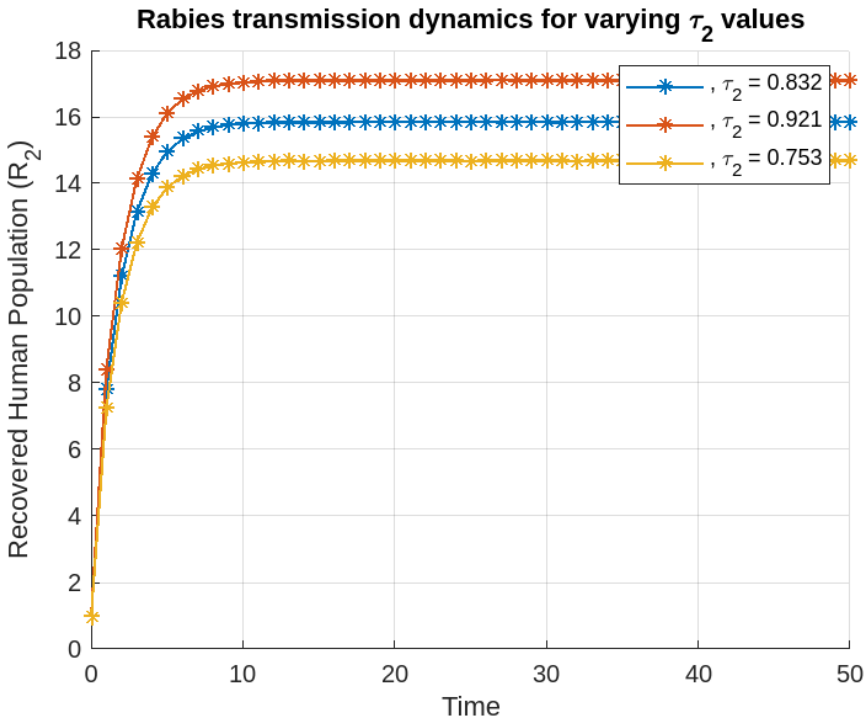

Figure 8. Population variation recovered (human) over time for fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25$, $\mu = 0.4$, $\pi_2 = 232432$, $S_2 = 132432$, $E_2 = 50000$, $I_2 = 40000$, $R_2 = 10000$, $\eta_2 = 0.623$, $\omega_2 = 0.251$, $\tau_2 = [0.832, 0.921, 0.753], \theta_2 = 0.432$.

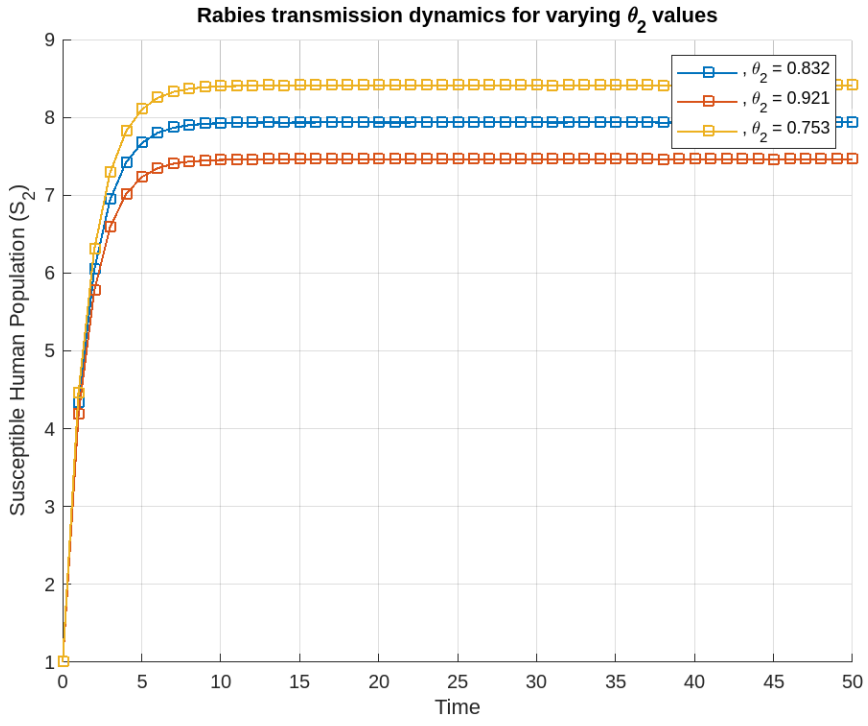

Figure 9. Population variation susceptible (human) over time for fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.4$, $\pi_2 = 232432, S_2 = 132432, E_2 = 50000, I_2 = 40000, R_2 = 10000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = [0.832, 0.921, 0.753]$.

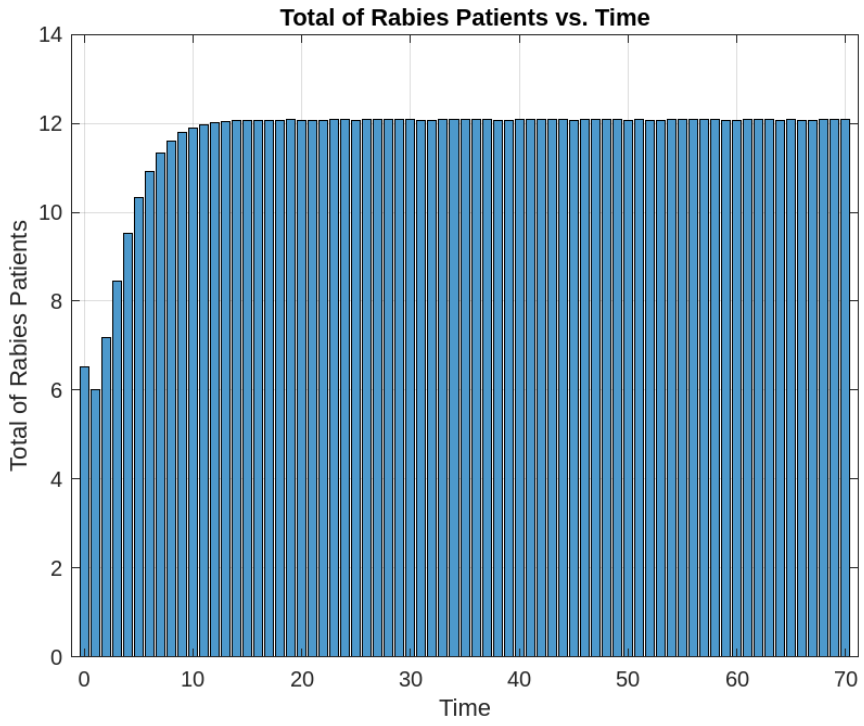

Figure 10. Population variation of Rabies Patients over time for fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.4, \pi_2 = 232432, S_2 = 132432, E_2 = 50000, I_2 = 40000, R_2 = 10000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.832$.

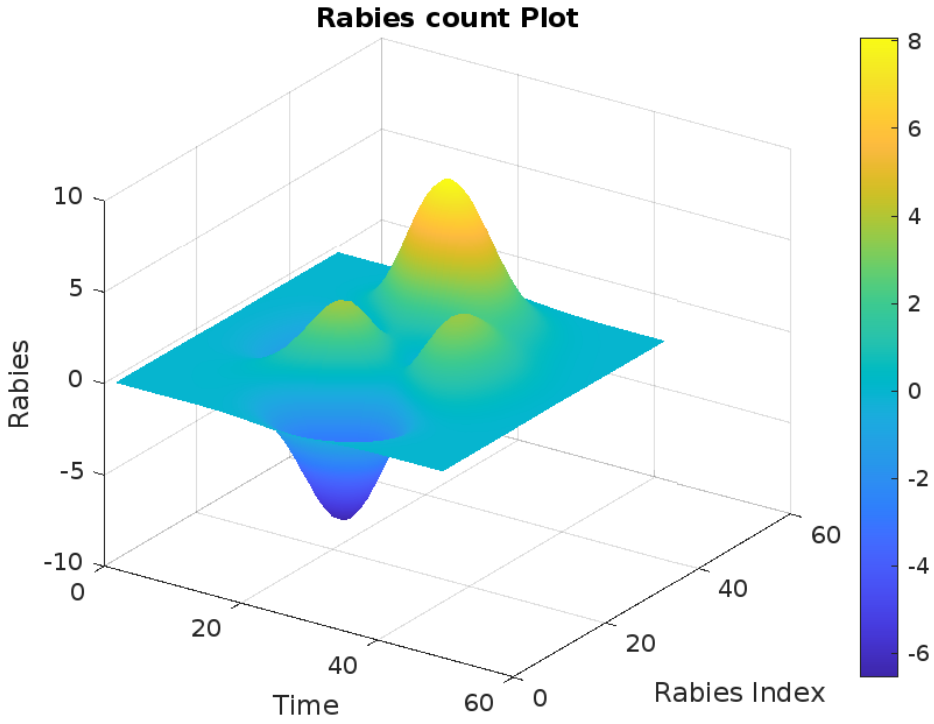

Figure 11. Rabies index over time for a range of fractional order parameters.

$\pi_1 = 125321, S_1 = 50000, E_1 = 30000, I_1 = 25321, R_1 = 20000, \alpha_1 = 1.5, \beta_1 = 0.75, \delta_1 = 0.83, \gamma_1 = 0.45, \lambda_1 = 0.143, \mu_1 = 0.25, \mu = 0.4, \pi_2 = 232432, S_2 = 132432, E_2 = 50000, I_2 = 40000, R_2 = 10000, \eta_2 = 0.623, \omega_2 = 0.251, \tau_2 = 0.432, \theta_2 = 0.832$.

8. Discussion

This work introduces a compartmental model for rabies transmission between dogs and humans using both classical and fractal fractional-order differential equations. The model includes the significant compartments of susceptible, exposed, infected, and recovered dog and human individuals, and accounts for the impact of contact tracing on the transmission dynamics. As a public health strategy, contact tracing is important for reducing transmission rates by tracking those who have come into contact with an infected person and checking on them.

The main aim of the study was to explore how the dynamics of the disease change over time with various fractions of the Caputo-Fabrizio derivative. The numerical simulations were prepared using the Adams-Bashforth method and compared the results of fractional orders of $\alpha_1 = [5.53, 3.28, 0.9]$, which is the classical level of analysis.

The numerical simulation results suggest the following: with decreasing fractional order (i.e., moving away from the classical case), the weight of memory becomes more apparent and is measurable in the additional day(s) until peak infection and the additional day(s) for the compromised compartments to trend downward. The Infected compartments for humans and dogs appear to linger longer when memory is factored into the modeling. The memory effect requires more robust modeling in fractional calculus, and the performance of the results suggests the power and importance of fractional-order operators in the modeling of epidemiological disease.

The basic reproduction number R_0 was calculated, and the disease-free and endemic equilibrium points were symbolically determined.

9. Conclusion

The research aims to develop a fractional-order mathematical model for the transmission of rabies in both humans and dogs by utilising the CFFROD idea and a two-step Adams-Bashforth numerical technique for simulation. The investigation encompassed susceptible, exposed, infected, and recovered populations as well as the dynamics of viral concentration over time for various fractional orders. The developed fractional-order mathematical model sheds light on the intricate dynamics of rabies transmission, emphasising the importance of early intervention tactics, thorough population dynamics analyses, and well-thought-out control measures in stopping the disease's spread and preserving the health of both human and animal populations.

Examining equilibrium points for both endemic and disease-free conditions revealed the reproduction number. R_0.The numerical simulation findings demonstrated that the number of susceptible dogs and humans grows abruptly and then stabilizes as the public health efficacy parameter values increase. Furthermore, as the number of infected dogs and humans continues to decline, there will be no more infections. This suggests that educating the public about the administration of vaccinations both before and after exposure, as well as responsible dog ownership, can significantly reduce the number of cases of rabies. As such, it is recommended that the public be made aware of rabies control and elimination to achieve the country's goal of having no rabies cases by 2030.

Author Contributions

Conceptualization, N.K.J. and A.M.; methodology, N.K.J. and A.M.; software, N.K.J., A.M., S.D., and C.M.; validation, N.K.J., A.M., S.D., and N.A.A.; formal analysis, N.K.J. and A.M.; investigation, N.K.J., A.M., and C.M.; resources, N.K.J., A.M., S.D., N.A.A., and C.M.; data curation, N.K.J. and A.M.; writing—original draft preparation, N.K.J. and A.M.; writing—review and editing, N.K.J., A.M., and N.A.A.; visualization, N.K.J., A.M., S.D., and N.A.A.; supervision, N.K.J., A.M., S.D., N.A.A., and C.M.; project administration, N.K.J., A.M., and C.M.; funding acquisition, N.K.J., A.M., S.D., and C.M. All authors have read and agreed to the published version of the manuscript.

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Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Informed consent was obtained from all subjects involved in the study.

Data Availability Statement

The data underlying the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no conflict of interest.

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