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Article

Efficient Algorithms for Solving Problems Coupled Oscillations of Complex Axially Symmetric Ring Lattices of Dielectric Resonators

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Abstract: Coupled oscillations of ring lattices with different types of dielectric resonators are considered. New analytical equations for complex frequencies and amplitudes of resonators, without restrictions on their number, are obtained. General analytical solutions for the frequencies and amplitudes of coupled oscillations for different ring lattices built on different resonators are found. It is noted that the obtained equations are also suitable for describing coupled oscillations of a ring lattices with degenerate oscillations of resonators, as well as with structures that contain ring lattices with different number elements. In general, the solutions for eigen waves propagating in periodic ring structures of DR are found. The solutions for several ring lattices consisting of two, three and four resonators of different types are compared with the numerical values found from the eigenvalues of the general coupling matrix. Good agreement between the analytical and the numerical results of calculating of the coupling matrix eigenvalues is demonstrated. The developed theory is the basis for the design and optimization of parameters of different devices of the microwave, theraherz and optical wavelength ranges, that built on a large number of dielectric resonators of various types. New equations obtained for calculating coupled oscillations of dielectric resonators also allow build more efficient models of scattering for optimization of various dielectric metamaterials.

Keywords: Dielectric Resonator; Coupled Oscillations; Ring Lattice; Circulant Matrix

1. Introduction

Today lattices of dielectric resonators (DR) find application in many different devices of the microwave, theraherce, infrared and optical wavelength ranges [1–4]. It's may be used in antennas [5–7]; filters [8–10]; sensors [11,12]; multiplexers [13]; modulators [14] and ather [15–21]. At the same time, many metamaterials differ from each other both in the shape and types of resonator oscillations, and in macroscopic characteristics–the structure and macroform of the lattices. Among them, ring lattices, characterized by the highest quality factor of coupled oscillations, are of considerable interest. Such lattices, as a rule, contain a very large number of elements, which significantly complicates optimization of their parameters.

The calculation of devices built on lattices of dielectric resonators more often based on the use of numerical methods. Numerical methods have great flexibility, but at the same time, they provide very little information about oscillatory processes and, in addition, require the use of significant computing resources. Therefore, the analysis of complex systems based on the use of Maxwell's equations and physical modeling continues to play an important role in understanding the behavior of electromagnetic fields in various structures.

In [22] analytical theory of scattering of electromagnetic waves on systems of dielectric resonators, based on the use of perturbation theory, is developed. The presented theory is based on the decomposition of electromagnetic scattering fields into coupled oscillations of complex systems of resonators. In this case, the central role is played by knowledge of the basis functions that display all possible lattice oscillations near a fixed frequency mode of an isolated partial resonator.

Obtaining the basis functions of coupled oscillations is a rather complex task, especially in cases when the number of resonators is very large. The lattices of dilectric resonators typically contain hundreds of elements, so constructing basis functions of their natural oscillations is a complex task, the solution of which presents significant difficulties. However, in some cases, the basis functions of many elemental structures can be calculated in analytical form.

The aim of this study is to obtaining analytical solutions of basis functions of coupled oscillation for ring lattices of different dielectric resonatjrs with big number elements.

Section 2 of this work provides the basic definitions and also presents the initial equations obtained earlier for describing the coupled oscillations of dielectric resonators.

Section 3 examines the coupled oscillations of a system of identical DRs placed in a simple ring lattice.

In Section 4, a general system of equations is derived for the amplitudes of coupled oscillations of resonators of a finite number of axially symmetric ring lattices that contain the same number of elements. The general properties of the obtained equations are investigated. An approximate solution to the system of equations is found, taking into account the coupling between adjacent ring sublattices. The solution of the equation system for an infinite periodic structure of identical ring lattices DR is given. Dispersion equations that determine the dependence of the complex amplitudes of the resonators on frequency are obtained.

In Section 5, using the equations found in Section 4, the conditions describing the natural oscillations of ring lattices with degenerate oscillations of resonators are discussed.

Section 6 discusses various types of symmetry in coupled ring sublattices that simplify the calculation of coupled DR.

Section 7 discusses the possibility of generalizing the results of the theory to describe the natural oscillations of ring lattices with different (multiple) numbers of resonators.

Section 8 provides examples of calculating the frequencies of various ring lattices, found by solving the system of equations derived in Section 4. The obtained results are compared with the data found from the calculation of the eigenvalues of the general coupling matrix.

The Conclusion notes a significant gain in the volume and speed of calculations of the frequencies and basis functions of complex ring structures of the DR, obtained by using the system of equations proposed in the work.

2. Coupled Oscillations of Dielctric Resonators

In the Trubin (2024) [22] it's looked for a solution to the common problem of coupled oscillations of a system consisting of N DRs. It was assumed that all isolated resonators have the same natural frequency ω_0, but they can have different sizes or be made of different dielectrics. A solution for the electromagnetic field $\left(\vec{e},\vec{h}\right)$ of N DR coupled oscillation was obtained in the form of an expansion on the fields of isolated resonators $\left(\vec{e}_s,\vec{h}_s\right)$ $(\mathrm{e_s}, \mathrm{h_s})$:

$$\begin{pmatrix}\vec{e}\\ \vec{h}\end{pmatrix} = \sum_{s=1}^{N} b_s \begin{pmatrix}\vec{e}_s\\ \vec{h}_s\end{pmatrix} \tag{1}$$

In general, using perturbation theory, was obtained linear homogeneous equation system for unknown amplitudes $\|b_s\|$ (1):

$$\sum_{s=1}^{N} \kappa_{st} b_s - \lambda b_t = 0; (s, t = 1, 2, \dots, N) \tag{2}$$

where

$$\lambda = \frac{2\,(\tilde{\omega} - \omega_0)}{\omega_0} = 2\left(\frac{\delta\omega}{\omega_0} + \frac{i\omega"}{\omega_0}\right) \tag{3}$$

$\tilde{\omega}$ - complex frequency of coupled oscillations of the DR system; ω_0- real part of the frequency of isolated DR; $\delta\omega = Re(\tilde{\omega} - \omega_0)$; $\omega" = Im(\tilde{\omega})$.

The distribution of the amplitudes of coupled oscillations of resonators $[\![b_s]\!]$ was formulated as an eigenvalue problem for a finite-dimensional coupling operator K= $\|\kappa_{st}\|$:

$$K = \begin{pmatrix} i\tilde{k}_1 & \kappa_{21} & \kappa_{31} & \dots & \kappa_{N,1} \\ \kappa_{12} & i\tilde{k}_2 & \kappa_{32} & \dots & \kappa_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \kappa_{1,N-1} & \kappa_{2,N-1} & \kappa_{3,N-1} & \dots & \kappa_{N,N-1} \\ \kappa_{1,N} & \kappa_{2,N} & \kappa_{3,N} & \dots & i\tilde{k}_N \end{pmatrix} \tag{4}$$

where $\kappa_{st} \neq \kappa_{ts}$ - are mutual coupling coefficients of a s–th and t–th different DR defined as Trubin (2024) [22]:

$$\kappa_{st} = \frac{i}{2\omega_0 w_t (1 + \delta_{st})} \oint \left\{ \left[\vec{e}_s, \vec{h}_t^*\right] + \left[\vec{e}_t^*, \vec{h}_s\right] \right\} \vec{n} ds,$$

where; $\vec{n}$ - normal to surface s_t of the t- th DR, and w_t - energy stored in a dielectric of the t- th resonator; δ_{st}- Kronecker symbol, $i = \sqrt{-1}$; $*$ - complex conjugate symbol.It has been shown by Trubin (2024) [22], that diagonal elements of the matrix K is determined only by the magnitude of the radiation of s - th partial resonators, represented by coupling coefficients $\tilde{k_s}$ with external structure ($s = 1, 2, \dots, N$):

$$\tilde{k}_s = 1/Q_s^{\Sigma},$$

Q_s^{Σ} - the radiation quality factor of the s-th resonator into the external structure.

Equating to zero, the determinant of the system (2), was obtained the characteristic equation, the solution of which determines the complex frequency splitting that arises due to the electromagnetic influence of the resonators. In a case of non-degenerate oscillations, to each value of the frequency $\tilde{\omega^s} = \omega^s + i\omega^{s''}$ ($s = 1, 2, \dots, N$) corresponds its own column vector:

$$[\![b_t^s]\!] = \begin{pmatrix} b_1^s \\ b_2^s \\ \vdots \\ b_N^s \end{pmatrix} \qquad (s, t = 1, 2, \dots, N) \tag{5}$$

of the coupling operator K (4), determining the distribution of amplitudes of partial resonators. Thus, in the absence of degeneracy, a system consisting of N resonators is characterized by a $N \times N$ matrix of amplitudes:

$$B = \begin{pmatrix} b_1^1 & b_1^2 & \dots & b_1^N \\ b_2^1 & b_2^2 & \dots & b_2^N \\ \vdots & \vdots & \vdots & \vdots \\ b_N^1 & b_N^2 & \dots & b_N^N \end{pmatrix} \tag{6}$$

In the general cases, the solution of the system (2) is carried out numerically, but, in some cases can be found in analytical form, which significantly simplified and increases the speed of calculations, especially for large DR systems ($N \gg 1$).

In this study we will find solutions to the system of Equation (2) in analytical form for different ring lattices, containing DR of different types.

3. Coupled Oscillations of One Ring Lattice of Identical Dielctric Resonators

At beginning we have considered particular solutions of (2) for a simple ring lattice of identical DRs. In this case we assumed that all coupling coefficients of the resonators with the external structure are equal to each other. $\tilde{k}_s = \tilde{k}_0$ ($s = 1, 2, \dots, N$). We assumed that each resonator, (indicated by a circle in **Figure 1a**), is located at the vertex of a regular polygon.

It's also assumed that the coupling coefficients of the identical resonators satisfy conditions to the symmetry: $\kappa_{st} = \kappa_{ts}$ and translation: $\kappa_{st} = \kappa_{|s-t|}$ ($s, t = 1, 2, \dots, N - 1$).

The last condition can also be rewritten in the form: $\kappa_{st} = \kappa_v = \kappa_{-v}$, where $v = |s - t|$. In this case, matrix (4) becomes a circulant matrix [23,24]; the elements of each row are obtained by cyclically permuting the elements of the previous one.

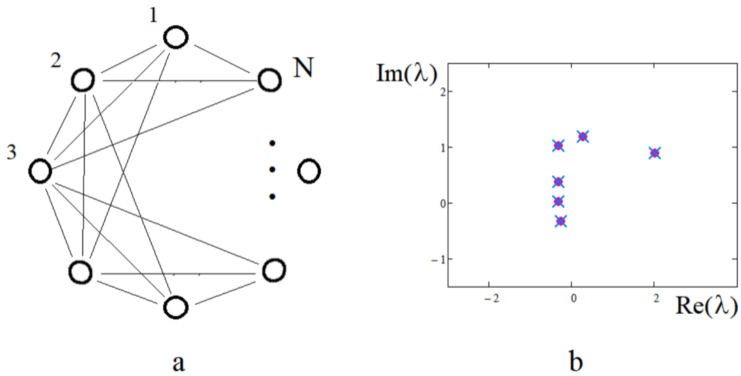


Figure 1. Ring lattices of N identical DRs (**a**). All dielectric resonators are indicated by circles; the coefficients of mutual coupling between them are indicated by straight lines connecting the resonators. Distribution of complex eigenvalues of the coupling matrix (4). (**b**) for lattice with $N = 11$ DR; the coupling coefficients of the resonators with external structure: $\tilde{k}_0 = 0,5$; mutual coupling coefficients: $\kappa_1 = 0,3 + 0,3i$; $\kappa_2 = 0,25 + 0,1i$; $\kappa_3 = 0,2 - 0,1i$; $\kappa_4 = 0,15 - 0,2i$; $\kappa_5 = 0,1 + 0,1i$. (Here and below, the numerical values of the coupling coefficients was taken arbitrarily). The results of comparison of the eigenvalues of 11 DR ring structure, found numerically (points) and calculated using formula (7) (crosses).

The eigenvalues of such a circulant matrix are well known [23,24]. For the defined above coupling operator (4):

$$\lambda^j = i\tilde{k}_0 + \sum_{v=1}^{N-1} \kappa_v (\eta_j)^v \tag{7}$$

where

$$\eta_j = \exp(\frac{2\pi i}{N} j); \qquad (j = 0, 1, \dots, N-1)$$

is the j-th complex N-th root of unity.

The matrix of normalized eigenvectors of a circulant matrix has the form [24]:

$$\vec{b}_o^j = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \eta_j^1 \\ \vdots \\ \eta_j^{N-1} \end{pmatrix} \tag{1}$$

or from (6):

$$B = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \eta_1 & \cdots & \eta_{N-1} \\ 1 & \eta_1^2 & \cdots & \eta_{N-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \eta_1^{N-1} & \cdots & \eta_{N-1}^{N-1} \end{pmatrix} \tag{2}$$

As follows from (7), (3), the frequencies of natural oscillations of regular ring lattices of identical DRs are linear functions of the coupling coefficients, and the eigenvectors (8, 9) do not depend on the coupling coefficients at all, but are determined only by the number of resonators in the structure.

Figure 1b shows the result of comparison of the eigenvalues of the ring structure of 11 DR, found numerically (points) and calculated using formula (7) (crosses).

4. Coupling Oscillations of Several Ring Lattices of Different Dielctric Resonators

Axially symmetric ring lattices with the same number of elements, each of which contains DR of different types, allows an analytical description of coupled oscillations in a general form.

In the figures below we designated **M** ring lattices of different DR. The first lattice were designated by the number **1** ; the second lattice by the number **2** and the Mth lattice by the number **M**. In all the figures, resonators of different types were indicated by circles of different sizes. To improve perception, all resonators of one sublattice were connected by lines.

The equations describing the coupled oscillations of ring lattices of resonators of different types are also described by the same system of Equation (2), but we need taking into account the asymmetry of the coupling coefficients between resonators of different types: $\kappa_{st}^{uw} \neq \kappa_{ts}^{uw}$ if $u \neq w$. We used indices in the upper part of the coupling coefficients to denote the numbers of the ring lattices, and indices in the lower part of the coupling coefficients to denote the numbers of the resonators in the structure. In this case, $s \in u$ th lattice and $t \in w$ th lattice.

We also proposed that each ring lattice contains the same number of resonators denoted N. All sublattices are located axially symmetrically relative to the allocated common axis. Moreover, they may not be located in the same spatial plane.

As follows from (8), all eigenvectors of isolated ring lattices are the same, however, for ease of perception, we have designated each of the vectors of the w-th lattice by:

$$\vec{b}_o^j = \vec{b}_o^{wj} = \begin{pmatrix} b_1^{wj} \\ b_2^{wj} \\ \vdots \\ b_N^{wj} \end{pmatrix} \tag{10}$$

$(w = 1, 2, \dots, M; j = 0, 1, \dots, N-1)$

The j-th eigenvector (7) of the coupling operator of the w-th isolated ring lattice obeys the equation:

$$\left(i\tilde{k}_w - \lambda_o^{wj}\right) b_o^{tj} + \sum_{n=1, n\neq t}^{N-1} \kappa_{nt}^{w} b_o^{nj} = 0 \tag{11}$$

$(w = 1, 2, \dots, M)$

An axially symmetric arrangement of sublattices with the same number of resonators allows reducing the total number of mutual coupling coefficients. We redefined the "vector" of mutual coupling between the s-th and w-th ring sublattices, taking into account the rotational symmetry of the structure, as well as the condition using: $\kappa_{tu}^{sw} = \kappa_{t-u}^{sw}$:

$$\vec{K}_{uw} = \begin{pmatrix} \kappa_0^{uw} \\ \kappa_1^{uw} \\ \vdots \\ \kappa_{N-1}^{uw} \end{pmatrix} \tag{12}$$

and also redetermined the mutual coupling vector of the resonators of the w-th sublattice:

$$\vec{K}_w = \begin{pmatrix} \kappa_1^{w} \\ \kappa_2^{w} \\ \vdots \\ \kappa_{N-1}^{w} \end{pmatrix} \tag{13}$$

We isolated and regrouped the terms in Equation (2), describing to the oscillations of the $N \times M$ resonators, in the form of partial sums, each of which relates to the oscillations of the selected w-th and other sublattices:

$$\sum_{s=1}^{N\times M} \kappa_{st} b_s - \lambda b_t = \left(i\tilde{k}_w - \lambda\right) b_t + \sum_{n=1, n\neq t}^{N-1} \kappa_{nt}^{w} b_n + \sum_{u\neq w=1}^{M} \sum_{v=0}^{N-1} \kappa_{vt}^{uw} b_v = 0 \tag{14}$$

where, $t = 1, 2, \dots, N \times M; w = 1, 2, \dots, M$.

The solution of system (14) for the j-th type of coupled oscillations of the resonators of the w-th sublattice was sought in the form:

$$\vec{b}^{(j)} = a^w \vec{b}_o^j \tag{15}$$

Thus, in fact we assumed that the amplitude distribution in a complex coupled structure of resonators preserves the amplitude distribution characteristic of the natural oscillations of isolated ring sublattices (9). The presence of resonators of other sublattices, in the accepted conditions of symmetry, changes only the amplitudes a^w of the coupled oscillations.

Substituting (15) into (14), and then multiplied (11) by a^w and subtracted it from (14) we obtained:

$$\left(\lambda_o^{wj} - \lambda\right) a^w b_o^{tj} + \sum_{u \neq w=1}^{M} \left[\sum_{v=0}^{N-1} \kappa_{vt}^{uw} b_o^{(v)}\right] a^u = 0 \tag{16}$$

Since the summation $\sum_{v=0}^{N-1} \kappa_{vt}^{uw} b_o^{(v)j}$ was carried out over all resonators of the u-th sublattice, it does not depend on the initial values of the indexes:

$$\sum_{v=0}^{N-1} \kappa_{vt}^{uw} b_o^{(v)j} = \sum_{v=0}^{N-1} \kappa_{(t+v)t}^{uw} b_o^{(t+v)j},$$

The replacement of variables under the summation sign corresponds to a simple rearrangement of the terms, so system (16) can be represented as:

$$\left(\lambda_o^{wj} - \lambda\right) a^w b_o^{tj} + \sum_{u \neq w=1}^{M} \left[\sum_{v=0}^{N-1} \kappa_{(t+v)t}^{uw} b_o^{(t+v)t}\right] a^u = 0 \tag{17}$$

Dividing (17) by b_o^{tj} and took into account (8):

$$\frac{b_o^{(t+v)j}}{b_o^{tj}} = \sqrt{N} b_o^{(v)j}$$

As a result, we obtained a system of equations that does not explicitly depend on the resonators,

$$\left(\lambda_o^{wj} - \lambda\right) a^w + \sqrt{N} \sum_{u \neq w=1}^{M} \left[\sum_{v=0}^{N-1} \kappa_v^{uw} b_o^{(v)j}\right] a^u = 0 \tag{18}$$

$(w = 1, 2, \dots, M)$

connecting only the amplitudes of the sublattices. Here we have again used the condition: $\kappa_{qv}^{uw} = \kappa_{q-v}^{uw}$.

The system of Equation (18) can be rewritten in a more compact form, taking into account the representation (12):

$$\left(\lambda_o^{wj} - \lambda\right) a^w + \sqrt{N} \sum_{u \neq w=1}^{M} \left(\vec{K}_{uw}, \vec{b}_o^j\right) a^u = 0 \tag{19}$$

$(w = 1, 2, \dots, M)$

Taking into account the definitions (3), we also obtained, $\lambda_o^{wj} - \lambda = 2(\omega_o^{wj} - \omega)/\omega_0$. Where ω_o^{wj} – frequency of the j-th natural oscillation of the w-th ring sublattice. From which it follows that the system of Equation (19) determines the frequency detuning of the sublattices when they interact with each other.

In total, the obtained systems of M Equation (18), or (19), in general form determines the frequencies and amplitudes of coupled oscillations of M axially symmetric ring sublattices with the same number of resonators.

It is interesting that the coupling vectors of the resonators "inside" the ring sublattices (13) are not included in the system (18) in an explicit form. They determine the frequencies of natural oscillations of the sublattices λ_o^{wj}.

By analogy with the oscillations of individual isolated resonators (4), we can introduce into consideration the coupling matrices of the ring sublattices for each j-th oscillation:

$$K^{(j)} = \begin{pmatrix} \lambda_o^{1j} & \sqrt{N}\left(\vec{K}_{21}, \vec{b}_o^j\right) & \cdots & \sqrt{N}\left(\vec{K}_{M1}, \vec{b}_o^j\right) \\ \sqrt{N}\left(\vec{K}_{12}, \vec{b}_o^j\right) & \lambda_o^{2j} & \cdots & \sqrt{N}\left(\vec{K}_{M2}, \vec{b}_o^j\right) \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{N}\left(\vec{K}_{1M}, \vec{b}_o^j\right) & \sqrt{N}\left(\vec{K}_{2M}, \vec{b}_o^j\right) & \cdots & \lambda_o^{Mj} \end{pmatrix} \tag{20}$$

the eigenvalues and eigenvectors of which, together with (7), (8), (15), determine the frequencies and amplitudes of the structure.

In contrast to the natural oscillations of individual resonators (2), the diagonal elements of the coupling matrix (20) λ_o^{wj} take on complex values due to the accumulated energy in each sublattice.

The physical meaning of the dependence of matrix elements $K^{(j)}$ on the type of oscillations j is explained by the different distribution of sublattice fields, which determines their interaction with each other.

The "interaction" of the u-th and w -th sublattices is expressed by a "potential function" $\sqrt{N}\sum_{u\neq v=1}^{M}\left(\vec{K}_{uw},\vec{b}_o^j\right)$, determinable by the set of all coupling coefficients and shapes of ring structures.

If the interaction between sublattices $\vec{K}_{uw}=0$, the eigenvalues of the $K^{(j)}$ matrix are determined only by the set of eigenvalues of the isolated ring sublattices λ_o^{wj} (7).

In the case of identical resonators in all sublattices $\lambda_o^{uj}=\lambda_o^j$, the eigenvalues of matrix (20) are determined by the expressions:

$$\lambda^j=\lambda_o^j+\overline{\lambda}^j \tag{21}$$

where $\overline{\lambda}^j$ - eigenvalues of a matrix:

$$\overline{K}^{(j)}=\begin{pmatrix} 0 & \sqrt{N}\left(\vec{K}_{21},\vec{b}_o^j\right) & \cdots & \sqrt{N}\left(\vec{K}_{M1},\vec{b}_o^j\right) \\ \sqrt{N}\left(\vec{K}_{12},\vec{b}_o^j\right) & 0 & \cdots & \sqrt{N}\left(\vec{K}_{M2},\vec{b}_o^j\right) \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{N}\left(\vec{K}_{1M},\vec{b}_o^j\right) & \sqrt{N}\left(\vec{K}_{2M},\vec{b}_o^j\right) & \cdots & 0 \end{pmatrix} \tag{22}$$

It follows that the appearance of other sublattices perturbs the eigenvalues λ_o^j of the coupling matrix. The magnitudes of this perturbations are determined by the eigenvalues of the matrix $\overline{K}^{(j)}$.

An approximate solution to the system of Equation (19) can be found in analytical form, for example, for the case of M identical ring sublattices located equidistantly on the z axis of a cylindrical coordinate system (**Figure 2**). If we take into account the interaction of resonators of only neighboring sublattices, matrix (22) becomes tridiagonal. In this case, for identical resonators: $\vec{K}_{21}=\vec{K}_{12}$.

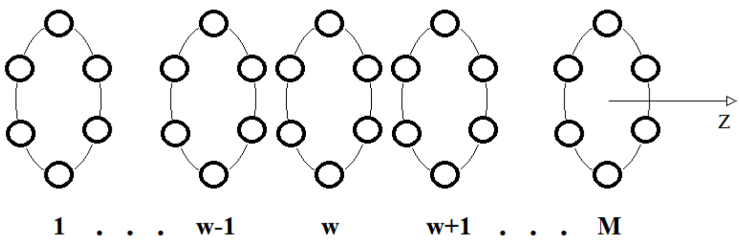


Figure 2. *M* identical ring sublattices.

For a finite number of sublattices, we will seek the solution of system (19) in the form:

$$a^w=a_0\begin{pmatrix} sin \\ cos \end{pmatrix}(\gamma w) \tag{23}$$

where a_0 and γ are constant do not depend on the sublattice numbers.

Substituting (23) into (19), after simple transformations and reduction of amplitudes a_0, we find:

$$\lambda^j=\lambda_o^j+2cos(\gamma)\sqrt{N}\left(\vec{K}_{12},\vec{b}_o^j\right) \tag{24}$$

Supplementing obtained expressions with conditions of symmetry of the sublattice amplitudes distribution: $|a^s|=\left|a^{M-s+1}\right|$, from (23) obtaine:

$$\left|\begin{matrix} sin \\ cos \end{matrix}(\gamma s)\right|=\left|\begin{matrix} sin \\ cos \end{matrix}[\gamma(M-s+1)]\right|$$

The solutions to these equations are:

$$\gamma = \gamma^s = \frac{s\pi}{(M+1)} \tag{25}$$

where $s = 1, 2, \dots, M$.

Conditions (25) together with (24), (7) determine the $N \times M$ frequencies of coupled oscillations of resonators in the taking into account approximations.

When describing a ring structure consisting of an infinite number of sublattices, the solution to the system of Equation (19) in the same approximation is represented in the form:

$$a^w = a_0 e^{\pm i\gamma w} \tag{26}$$

Here the dimensionless parameter $\gamma = \Gamma\Delta z$ has the meaning of the product of the wave number of the line Γ and the distance between adjacent sublattices Δz. We also note that the longitudinal distance between the centers of the lattice resonators is implicitly included in the coupling vectors $\vec{K}_{12}$.

Substituting (26) into (19), taking into account the approximations made above, we obtained the same dispersion Equation (24), in which now γ takes on continuous complex values depending on the detuning (3) relative to the frequencies of the natural oscillations of the sublattices and resonators. After simple transformations, taking into account definitions (3), we find:

$$\frac{\omega(\gamma)}{\omega_0} = \frac{\omega_0^j}{\omega_0} + \sqrt{N}\left(\vec{K}_{12}, \vec{b}_o^j\right) cos(\gamma) \tag{27}$$

where ω_0^j - frequencies of coupled oscillations of sublattices; ω_0 - natural frequency of isolated resonators; ω - real frequency.

Taking into account the accepted dependence on time, proportional to $e^{i\omega t}$, Equation (27) must be supplemented with the condition: $Im(\gamma) \leq 0$, corresponding to the requirement of decreasing resonator amplitudes due to radiation energy losses.

If the coupling coefficients between sublattices are $\vec{K}_{12} \to 0$, then $\omega(\gamma) \to \omega_0^j$; effective traveling waves of this type with low attenuation can exist only in the region of sublattices resonances. In general, in such structures, the dependence $\gamma(\omega)$ is characterized by strong dispersion, arising in systems of this class due to the existence of a large number of resonances in relatively narrow frequency bands.

5. Coupled Oscillations of Ring Lattice with Degenerate Oscillations of Resonators

The developed theory is also suitable for describing coupled oscillations of ring lattices containing resonators with degenerate types of natural oscillations.

Let us we have one ring lattice consisting of N identical resonators, in each of which M degenerate natural oscillations in frequency may be excited. In this case, $\vec{K}_w$ (12) can be interpreted as a vector of mutual coupling coefficients of a ring lattice, corresponding to the selected w th type of degenerate oscillations, at that the vectors b_o^{wj} have meaning of amplitude distributions of the resonators, also excited only by the wth type of the selected natural oscillations; $\vec{K}_{uw}$ (13) can be interpreted as a vector of mutual coupling between the resonators when degenerate natural oscillations of different types are excited in them.

As follows from (12), if the degenerate oscillations are orthogonal, then $\kappa_0^{uw} = 0$ for all , u, w, but $\kappa_s^{uw} \neq 0$, if $s > 0$ and $u \neq w$.

It is obvious that a similar approach also can be used to solve problems of natural oscillations of more complex ring structures, consisting of several ring lattices of different types of DR, each of which is characterized by degenerate types of oscillations.

6. Coupling Oscillations of Symmetrical Ring Lattices of Different Dielctric Resonators

When calculating the coupled oscillations of DR lattices, having the shape of a regular polygons, additional planes of symmetry may appear. In this case, it becomes possible to reduce the size of the Equation (19), by presenting them in a more convenient form.

Figure 3 shows two variants of the possible emergence of planes of symmetry with the axial placement of various ring lattices of the DR in the form of regular polygons.

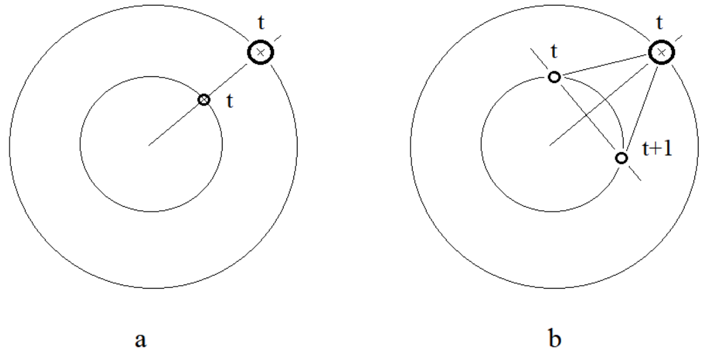


Figure 3. Possible planes of symmetry arising in a different ring sublattices of regular shape, located symmetrically with respect to a given angle of rotation of the structure in a cylindrical coordinate system. Resonators of different ring sublattices are arranged in pairs at equal angles (**a**). The resonators of one of the sublattice are located in the middle plane between the resonators of the other sublattice (**b**).

In the first case (**Figure 3a**) the angular coordinates of the corresponding t th resonators of the lattices are coincide.

In the second case, the angular coordinates of the resonators of one lattice lie exactly in the middle of the angular coordinates of the resonators of other lattice.

At that, in general, cases of simultaneous placement of lattices with two types of the indicated symmetries are also possible. For all the listed cases, it is possible to derive a system of equations suitable for calculating the frequencies of coupled oscillations.

To derive such a system, let us return to Equation (16).

In the case of the arrangement of the lattices shown in **Figure 3**, a, in the system of Equation (16) we have identified the terms related to the resonators, located symmetrically with respect to the plane passing through the center of the t th DR:

$$\sum_{v=0}^{N-1} \kappa_v^{uw} b_o^{(v)} = \sum_{v\in\Xi'} (\kappa_{t+v}^{uw} b_o^{(t+v)} + \kappa_{t-v}^{uw} b_o^{(t-v)}) = \sum_{v\in\Xi'} \kappa_{t+v}^{uw} (b_o^{(t+v)} + b_o^{(t-v)}) \quad (28)$$

where $\Xi' = 0, 1, 2, \dots, (N-1)/2$ for odd N number; $\Xi' = 0, 1, 2, \dots, \left(\frac{N}{2}\right)$ for an even N number of resonators.

Next we again used equality (8), from which we obtained:

$$b_o^{(t+v)j} + b_o^{(t-v)j} = 2cos(\frac{2v\pi j}{N}) b_o^{tj} \quad (29)$$

Substituting (29) into (28) and then into (16), after canceling out by b_o^{tj}, we again obtain a system of equations containing only the amplitudes of the sublattices:

$$\left(\lambda_o^{wj} - \lambda\right) a^w + 2\left[\sum_{u\neq w=1}^{M} \sum_{v\in\Xi'} \Delta_v \kappa_v^{uw} cos(2v\pi j/N)\right] a^u = 0 \quad (30)$$

here $w = 1, 2, \dots, M$; $\Delta_v = 1/(1 + \delta_{v,0} + \delta_{v,N/2})$; $\delta_{n,m}$ - Kronecker symbol.

The resulting system (30) is equivalent to the system of Equation (19) found above. This can be easily verified by performing inverse transformations of the sum in (30), replacing $cos(z) = 1/2\,[\exp(z) + \exp(-z)]$ and taking into account the possible number of resonators in the sublattices.

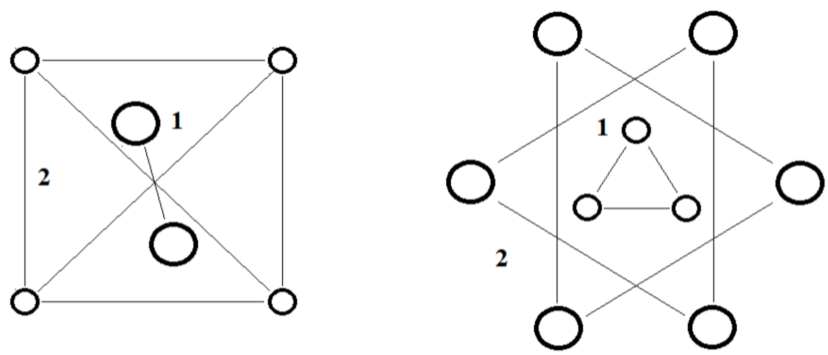
The advantage of Equation (30) is the smaller number of operations required to achieve a given accuracy, which sometimes becomes critical when optimizing complex structures with a large number of sublattices.

7. Coupling Oscillations of Ring Lattices with Different Number of Dielectric Resonators

It is not difficult to see, that the theory of coupled oscillations of axially symmetric ring lattices with the same number of resonators, developed above, can be easily generalized to structures containing axially symmetric lattices with different numbers of DRs. The above condition is satisfied by ring structures, each of which contains a number of resonators that is a multiple of the minimum of one of the components of the ring lattices.

Figure 4 shows several simple examples of constructing such lattices, each of which built from ring sublattices with different numbers of DR. In the first case (**Figure 4a**) the second ring sublattice consists of 2 two-element sublattices. The coupled oscillations of such a system can be described by Equation (19) with $N = 2$ and $M = 3$.

In the second example (**Figure 4b**) the second ring sublattice consists of 3 three-element sublattices. The coupled oscillations of this structure are also described by Equation (19) with $N = 3$ and $M = 3$.



a b

Figure 4. Examples of complex structurs build on different ring lattices with a not equal number of DR. Ring lattice consisting of 6 elements, composed of 3 sublattices: 2, 2 and 2 DR (**a**). Ring lattice consisting of 9 elements, composed of 3 sublattices 3, 3 and 3 DR (**b**).

In general, the description of ring lattices with an arbitrary unequal number of resonators using the system of Equation (19) is not correct, since in this case the structure does not have any rotational symmetry.

8. Examples

We have given several simple examples of calculating the parameters of coupled oscillations of ring lattices with different DR, using obtained above Equation (19). In all the examples given, the coupling coefficients of the resonators were chosen arbitrarily, in the form of sets of random complex numbers. In reality, for resonators of specific shapes and types of natural oscillations, they are calculated through the resonator fields, taking into account the specific values of their coordinates in the lattices and other parameters [22].

Figure 5b show the results of comparison of the eigenvalues of the coupling matrices calculated numerically (dots) from (4) with the values obtained from matrices (19), or (20) (crosses) for three different sublattices consisting of three different types of resonators.

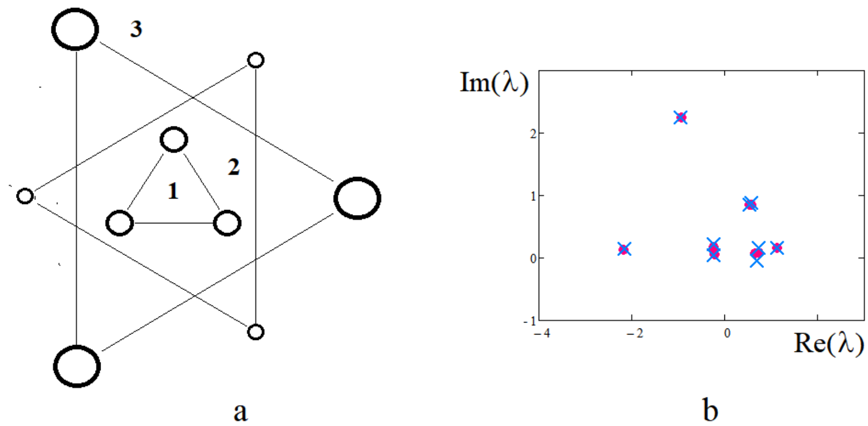


Figure 5. $M = 3$ Ring lattices of $N = 3$ different DR (**a**). The resonators of different types are indicated by different size circles. Distribution of complex eigenvalues of the coupling matrics (**b**): for the coupling coefficients of the DR of first sublattice $\tilde{k}_1 = 0,55$:; of the DR of the second sublattice $\tilde{k}_2 = 0,25$:; of the DR of the third sublattice $\tilde{k}_3 = 0,75$; mutual coupling coefficients of the DR of the first sublattice $\kappa_1^1 = \kappa_2^1 = -0,5 - 0,3i$:; of the DR of the second sublattice $\kappa_1^2 = \kappa_2^2 = 0,25 + 0,2i$; of the DR of the third sublattice $\kappa_1^3 = \kappa_2^3 = -0,75 + 0,6i$; vector of mutual coupling coefficients between resonators first and the second sublattice: $\vec{K}_{12} = \begin{pmatrix} -0,35 + 0,2i \\ -0,25 + 0,3i \\ -0,2 + 0,15i \end{pmatrix}$; between the resonators of the first and third sublattices $\vec{K}_{13} = \begin{pmatrix} 0,6 + 0,5i \\ 0,55 + 0,3i \\ 0,4 + 0,2i \end{pmatrix}$; between the resonators of the second and third sublattices $\vec{K}_{23} = \begin{pmatrix} -0,4 + 0,2i \\ -0,15 + 0,15i \\ -0,3 - 0,1i \end{pmatrix}$; vector of mutual coupling coefficients between resonators second and the first sublattice: $\vec{K}_{21} = \begin{pmatrix} -0,25 + 0,3i \\ -0,15 + 0,1i \\ -0,1 + 0,15i \end{pmatrix}$; between resonators third and the first sublattice: $\vec{K}_{31} = \begin{pmatrix} 0,4 + 0,3i \\ 0,35 + 0,1i \\ 0,25 + 0,1i \end{pmatrix}$; between resonators third and the second sublattice: $\vec{K}_{32} = \begin{pmatrix} -0,2 + 0,4i \\ -0,1 + 0,3i \\ -0,15 + 0,15i \end{pmatrix}$. The results of the numerical calculation of eigenvalues of the matrix (4) are dots; the analytical, obtained from (19), (9), are crosses (**b**).

The arbitrariness of the rotation angles of each of the 3 sublattices relative to the common center was specified by randomness of choice of numerical values of the mutual coupling coefficients of vectors (12), (13) and also when calculating the eigenvalues of the matrix of coupling coefficients (4).

Figure 6c,d show the results of comparison of analytical and numerical calculations of coupling matrices eigenvalues for 3 ring lattices, consisting of 4 different resonators, also obtained from (4) as well as (19), (20), or (30). In this case, it was assumed that each of the sublattices is located symmetrically in rotation angles relative to the neighboring ones.

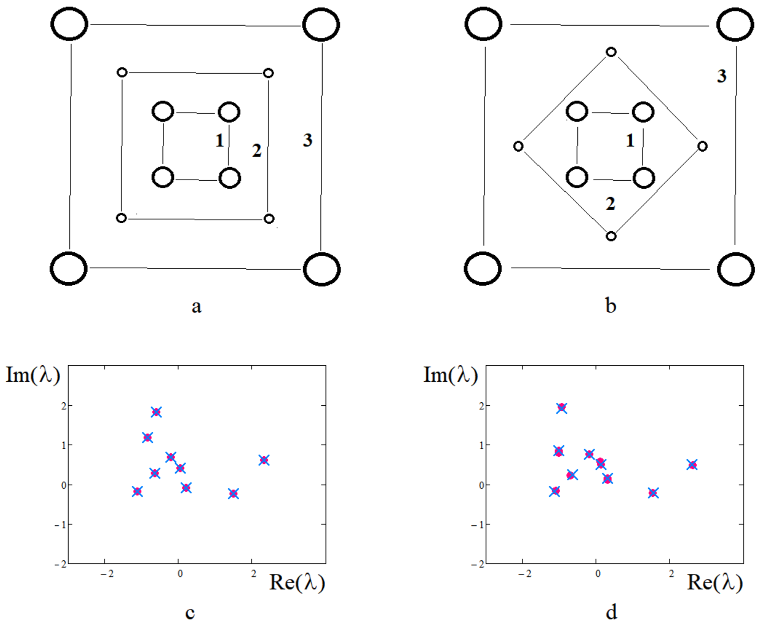


Figure 6. The ring structures of $M \times N$ of three different DR (**a**,**b**). Number of the DR in the one isolated sublattice: $N = 4$; the number of sublattices $M = 3$; the coupling coefficients of the resonators with external structure of the first ring sublattice: $\tilde{k}_1 = 0,5$; of the second sublattice: $\tilde{k}_2 = 0,25$; of the third sublattice: $\tilde{k}_3 = 0,75$; the mutual coupling coefficients vector of the first sublattice: $\vec{K}_1 = \begin{pmatrix} 0,5+0,25i \\ 0,15+0,1i \\ 0,5+0,25i \end{pmatrix}$; of the second sublattice: $\vec{K}_2 = \begin{pmatrix} 0,15-0,2i \\ 0,1+0,1i \\ 0,15-0,2i \end{pmatrix}$; of the third sublattice: $\vec{K}_3 = \begin{pmatrix} 0,65+0,3i \\ 0,35-0,2i \\ 0,65+0,3i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the first and the second sublattice: $\vec{K}_{12} = \begin{pmatrix} -0,3+0,3i \\ -0,2+0,2i \\ -0,1-0,1i \\ -0,2+0,2i \end{pmatrix}$; between resonators of the first and third sublattice: $\vec{K}_{13} = \begin{pmatrix} -0,4+0,3i \\ -0,2+0,1i \\ -0,1+0,15i \\ -0,2+0,1i \end{pmatrix}$; between resonators of the second and third sublattice: $\vec{K}_{23} = \begin{pmatrix} -0,6+0,5i \\ -0,4+0,3i \\ -0,15+0,1i \\ -0,4+0,3i \end{pmatrix}$; between resonators of the second and the first sublattice: $\vec{K}_{21} = \begin{pmatrix} -0,4+0,35i \\ -0,25+0,2i \\ 0,1+0,1i \\ -0,25+0,2i \end{pmatrix}$; between resonators of third and the first sublattice: $\vec{K}_{31} = \begin{pmatrix} -0,3+0,2i \\ -0,25+0,1i \\ 0,1-0,1i \\ -0,25+0,15i \end{pmatrix}$; between resonators of third and the second sublattice: $\vec{K}_{32} = \begin{pmatrix} -0,25+0,15i \\ -0,2+0,1i \\ -0,05-0,1i \\ -0,2+0,1i \end{pmatrix}$. The results of the numerical calculation of the eigenvalues of the matrix (4) are dots; the results of the analytical calculations obtaind from (19) are crosses (**c**,**d**).

Figure 7b shows the results of calculations of the frequencies of coupled oscillations of 4 lattices, each of which consists of 2 resonators of different types. In this case it was assumed that the lattices were rotated at unequal angles relative to a common center.

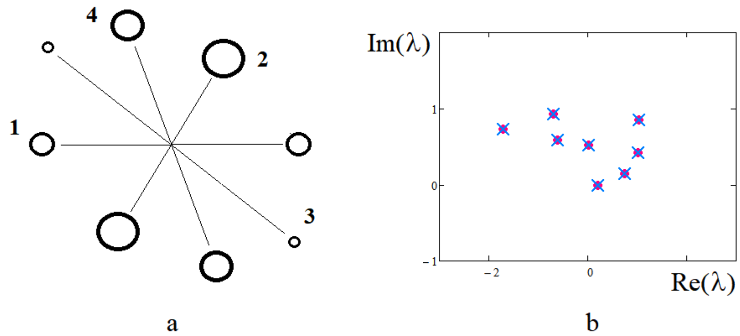


Figure 7. Ring lattice of $M \times N$ (**a**) different DR. (**b**): the number of resonators in the sublattice: $N = 2$; the number of the sublattices: $M = 4$; the coupling coefficients of the resonators with external structure of the first sublattice $\tilde{k}_1 = 0,5$: ; the coupling coefficients of the resonators with external structure of the second sublattice: $\tilde{k}_2 = 0,75$; the coupling coefficients of the resonators with external structure of the third sublattice: $\tilde{k}_3 = 0,75$; the coupling coefficients of the resonators with external structure of the fourth sublattice: $\tilde{k}_4 = 0,6$; mutual couling coefficient of the resonators of the first sublattice: $\kappa_1^1 = 0,35 - 0,2i$; mutual couling coefficient of the resonators of the second sublattice: $\kappa_1^2 = 0,7 + 0,3i$; mutual couling coefficient of the resonators of the third sublattice: $\kappa_1^3 = 0,1 - 0,2i$; mutual couling coefficient of the resonators of the fourth sublattice: $\kappa_1^4 = 0,4 + 0,2i$; vector of mutual coupling coefficients between resonators of the first and the second sublattice: $\vec{K}_{12} = \begin{pmatrix} -0,6 + 0,5i \\ -0,4 + 0,3i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the first and the third sublattice: $\vec{K}_{13} = \begin{pmatrix} 0,2 + 0,15i \\ -0,15 + 0,1i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the first and the fourth sublattice: $\vec{K}_{14} = \begin{pmatrix} 0,3 + 0,2i \\ -0,1 + 0,1i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the second and the third sublattice: $\vec{K}_{23} = \begin{pmatrix} 0,15 + 0,2i \\ -0,1 + 0,15i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the second and the fourth sublattice: $\vec{K}_{24} = \begin{pmatrix} 0,85 + 0,3i \\ -0,15 + 0,15i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the third and the fourth sublattice: $\vec{K}_{34} = \begin{pmatrix} 0,25 + 0,1i \\ -0,2 + 0,15i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the second and the first sublattice: $\vec{K}_{21} = \begin{pmatrix} -0,3 - 0,1i \\ -0,2 + 0,1i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the third and the first sublattice: $\vec{K}_{31} = \begin{pmatrix} 0,3 - 0,1i \\ -0,2 + 0,1i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the fourth and the first sublattice: $\vec{K}_{41} = \begin{pmatrix} 0,4 - 0,25i \\ -0,15 + 0,15i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the third and the second sublattice: $\vec{K}_{32} = \begin{pmatrix} -0,5 - 0,1i \\ -0,3 + 0,1i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the fourth and the second sublattice: $\vec{K}_{42} = \begin{pmatrix} 0,65 - 0,35i \\ -0,2 + 0,2i \end{pmatrix}$; vector of mutual coupling coefficients between resonators of the fourth and the third sublattice: $\vec{K}_{43} = \begin{pmatrix} 0,1 - 0,15i \\ -0,1 + 0,1i \end{pmatrix}$. The results of the numerical calculation of eigenvalues of the matrix (4) are dots; the analytical results obtained by using equation system (19) are crosses (**b**).

Figure 8 shows the results of calculating the natural frequencies of a simple ring lattice consisting of 3 resonators, each of which is characterized by 3 degenerate oscillations. It was assumed that the degenerate oscillations in each resonator are orthogonal to each other. The coupling coefficients of the resonators were also set arbitrarily, taking into account the symmetry of the structure, inequalities of the coefficients of mutual coupling for oscillations of different types, as well as the comments made in paragraph V.

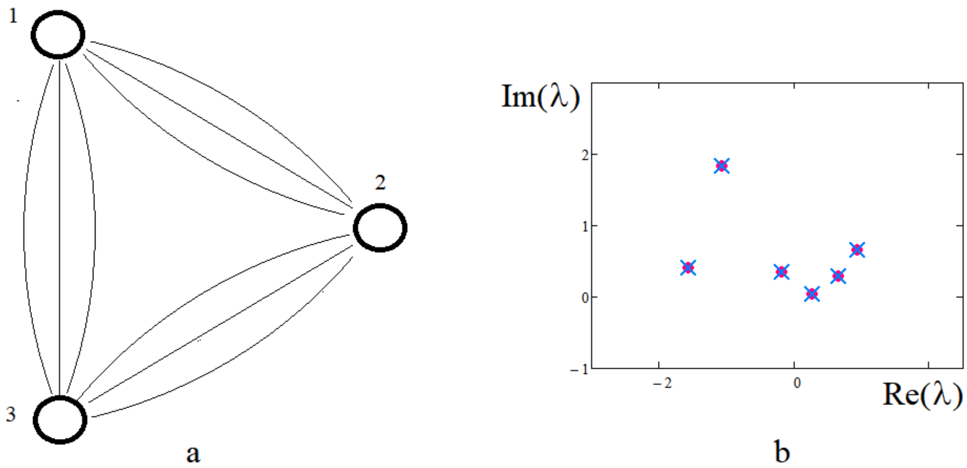


Figure 8. Ring lattice of $N = 3$ (**a**) identical DR with $M = 3$ degenerate oscillations. (**b**): the coupling coefficients of the resonators with external structure for the first degenerate oscillations: $\tilde{k}_1 = 0,55$; the coupling coefficients of the resonators with external structure for the second degenerate oscillations: $\tilde{k}_2 = 0,25$; the coupling coefficients of the resonators with external structure for the third degenerate oscillations: $\tilde{k}_3 = 0,75$; mutual coupling coefficients of the resonators with only the first degenerate oscillations: $\kappa_1^1 = \kappa_2^1 = -0,5 - 0,3i$; mutual coupling coefficients of the resonators only with the second type of degenerate oscillations: $\kappa_1^2 = \kappa_2^2 = 0,25 + 0,2i$; mutual coupling coefficients of the resonators only with the third type of degenerate oscillations: $\kappa_1^3 = \kappa_2^3 = -0,75 + 0,6i$; vector of mutual coupling coefficients between the first and the second degenerate oscillations of the resonators: $\vec{K}_{12} = \begin{pmatrix} 0 \\ -0,25+0,3i \\ -0,25+0,3i \end{pmatrix}$; vector of mutual coupling coefficients between the first and the third type degenerate oscillations of the resonators: $\vec{K}_{13} = \begin{pmatrix} 0 \\ 0,55+0,3i \\ 0,55+0,3i \end{pmatrix}$; vector of mutual coupling coefficients between the second and third type of degenerate oscillations of the resonators: $\vec{K}_{23} = \begin{pmatrix} 0 \\ -0,15+0,15i \\ -0,15+0,15i \end{pmatrix}$; vector of mutual coupling coefficients between the second and first degenerate oscillations of the resonators: $\vec{K}_{21} = \begin{pmatrix} 0 \\ -0,15+0,1i \\ -0,15+0,1i \end{pmatrix}$; vector of mutual coupling coefficients between third and the first degenerate oscillations of the resonators: $\vec{K}_{31} = \begin{pmatrix} 0 \\ 0,35+0,1i \\ 0,35+0,1i \end{pmatrix}$; vector of mutual coupling coefficients between third and second degenerate oscillations of the resonators: $\vec{K}_{32} = \begin{pmatrix} 0 \\ -0,1+0,3i \\ -0,1+0,3i \end{pmatrix}$. The results of the numerical calculation of the eigenvalues of the matrix (4) are dots; the analytical calculation results, obtained from matrix (19), are crosses (**b**).

9. Conclusions

The calculation of the distribution of amplitudes and frequencies of coupled oscillations of M ring lattices, each of which consists of N resonators, is reduced to the calculation of eigenvectors and eigenvalues of the coupling matrix (4), containing $(N \times M)^2$ elements. The use of analytical expressions for calculating the eigenvectors (8) and eigenvalues of circulant matrices (7) in this case, as well as the systems of Equation (19) obtained in this article, allows us to reduce the size of matrices for calculating the parameters of ring structures to the size of the M^2 elements.

For example, when calculating the parameters of natural oscillations of a lattice consisting of three ring sublattices with ten resonators, it is necessary to calculate the eigenvectors and eigenvalues of a matrix (4), consisting of 900 elements. Using the obtained system of Equation (19) allows us to reduce the size of the coupling matrix to 9 elements.

In the case of calculating the parameters of ring lattices constructed on the basis of using spherical resonators with whispering gallery modes [22], for example, with magnetic oscillations: H_{nml}, typical values of the meridian indexes are $n = 30$ and more. In this case, the number of degenerate modes of one resonator is equal to $2n+1 = 61$, and the number of possible oscillations in one ring lattice containing only 10 re sonators is $N = 610$. To calculate the parameters of a relatively simple structure containing only two ring lattices, it is necessary to calculate the eigenvalues and eigenvectors of the coupling matrix, containing the $1,488\times10^6$ elements. Reducing this number of elements even by four times, makes it possible to calculate the parameters of such gigantic matrix and significantly reduces computation time.

Thus, the analytical solutions found significantly simplify the calculation of the parameters of coupled oscillations of complex lattices of different DR and are the basis for constructing a more effective scattering theory constructed on decomposition of the solutions into eigenoscillations of the structure.

The developed theory of coupled oscillations of different types dielectric resonators demonstrates to us a new interesting pattern of interaction of ring lattices in complex ring structures. Placed relative to a selected spatial axis, such ring lattices can interact with each other without changing the distribution of the amplitudes of their coupled oscillations within each other.

A general system of equations that describes the coupled oscillations of ring sublattices with the same number of resonators can be also used to calculate the parameters of much complex systems of coupled ring lattices containing a multiple number of resonators.

It is shown that the found system of equations also allows us to describe in general form the coupled oscillations of ring, as well as many ring lattices with degenerate DR modes.

Good agreement between the results of analytical and the numerical results of calculating the eigenvalues of the different coupling matrix is demonstrated.

Developed theory can be used as the basis for the design and optimization of scattering parameters of many different devices of the microwave, theraherz and optical wavelength ranges, that build on ring lattices of different types of dilectric resonators.

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Institutional Review Board Statement

The study was conducted in accordance with the Declaration of Helsinki, and approved by the Institutional Review Board of the Educational and Research Institute of Telecommunication Systems of the National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute".

Data Availability Statement

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Conflicts of Interest

The author declare no conflict of interest.

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